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HYPOELLIPTIC LAPLACIAN AND TWISTED TRACE FORMULA

by Bingxiao LIU

Abstract. — We give an explicit geometric formula for the twisted orbital integrals using the method of the hypoelliptic Laplacian developed by Bismut. Combining with the twisted trace formula, we can evaluate the equivariant trace of the heat operators of the Laplacians on a compact locally symmetric space. In particular, we revisit the equivariant local index theorems and twisted L_2-torsions for locally symmetric spaces.

Résumé. — On donne une formule géométrique explicite pour les intégrales orbitales semisimples tordues du noyau de la chaleur sur un espace symétrique, en utilisant la méthode du laplacien hypoelliptique développée par Bismut. Alors en combinant avec la formule des traces tordue, on peut évaluer les traces équivariantes de l'opérateur de la chaleur du laplacien sur un espace localement symétrique compact. En particulier, on revisite les théorèmes de l'indice équivariant local et de la torsion L_2 équivariante pour les espaces localement symétriques.

1. Introduction

The purpose of this paper is to give an explicit geometric formula for the semisimple twisted orbital integrals associated with the Casimir operator on symmetric spaces, which extends an important result of Bismut for semisimple orbital integrals [9, Chapter 6]. The method that we use is the theory of hypoelliptic Laplacian developed by Bismut [9]. Here, we start with establishing a geometric formulation for the twisted orbital integral. Then we explain how to adapt Bismut's method to get our explicit formula. In the context of cyclic base change theory, we also exploit our formula by typical examples.

To explore the power of our formula, we use it to revisit the local equivariant index theorems for compact locally symmetric space, and especially,

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we exhibit the computations on the twisted orbital integrals using representation theory of compact Lie groups. In the last subsection, we also discuss briefly the equivariant real analytic torsion. For further study on this topic using our explicit formula, we refer to the author's paper [36].

Let us now give more details on the content of this paper.

1.1. Real reductive group and symmetric space

Let G be a connected real reductive Lie group ([29, §7.2]) with Lie algebra $\mathfrak{g}$, and let $\theta \in \mathrm{Aut}(G)$ be a Cartan involution. Let K be the fixed point set of θ in G. Then K is a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. The Cartan decomposition of $\mathfrak{g}$ associated with θ is given by

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{1.1}$$

Put $m = \dim \mathfrak{p}$, $n = \dim \mathfrak{k}$.

Let B be a G and θ-invariant nondegenerate bilinear symmetric form on $\mathfrak{g}$, which is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. Let $U\mathfrak{g}$ be the enveloping algebra of $\mathfrak{g}$, and let $C^{\mathfrak{g}} \in U\mathfrak{g}$ be the Casimir operator associated with B.

Let $X = G/K$ be the associated symmetric space. Then the projection $p : G \to X$ is a K-principal bundle. The bilinear form B induces a Riemannian metric g^{TX} on X with nonpositive sectional curvature. Let $d(\cdot\,,\cdot)$ denote the Riemannian distance on X.

If (E, ρ^E) is a unitary representation of K, then $F = G \times_K E$ is a Hermitian vector bundle on X. Moreover, $C^{\mathfrak{g}}$ descends to an elliptic operator $C^{\mathfrak{g},X}$ acting on $C^\infty(X, F)$. Our main object is to study the operator $\mathcal{L}^X$ acting on $C^\infty(X, F)$, which is defined as the sum of $\frac{1}{2}C^{\mathfrak{g},X}$ with an explicit real constant (Definition 4.3). For $t > 0$, let $\exp(-t\mathcal{L}^X)$ be the associated heat operator.

1.2. Twisted orbital integrals

We introduce the geometric characterization for semisimple elements. Let $\mathrm{Isom}(X)$ be the Lie group of isometries of X. If $\phi \in \mathrm{Isom}(X)$, set $d_\phi(x) = d(x, \phi(x))$, $x \in X$. As in [21], ϕ is called semisimple if d_ϕ reaches its infimum value m_ϕ in X, and ϕ is called elliptic if ϕ has fixed points in X. If ϕ is semisimple, let $X(\phi) \subset X$ be the minimizing set of d_ϕ, which is a convex submanifold of X.

In [9, Chapter 3], given a semisimple element $\gamma \in G$ (viewed as an isometry of X), $X(\gamma)$ is a symmetric space associated with $Z^0(\gamma)$, the identity component of the centralizer of γ. Then Bismut gave a geometric interpretation for the associated orbital integrals $\mathrm{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X)]$, so that they can be written as integrals along the fibres of the normal bundle $N_{X(\gamma)/X}$. This geometric formulation plays a central role in Bismut's approach to his explicit geometric formula of $\mathrm{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X)]$. Using Bismut's formula, Shen [45, 46] gave a full proof of the Fried conjecture for compact locally symmetric spaces, completing the work of Moscovici and Stanton [41].

In this paper, we extend Bismut's result to the case of twisted orbital integrals. Let Σ be the compact Lie group of the automorphisms of (G, B, θ). If $\sigma \in \Sigma$, let Σ^σ be the closure of the subgroup of Σ generated by σ. Put $G^\sigma = G \rtimes \Sigma^\sigma$, $K^\sigma = K \rtimes \Sigma^\sigma$. We do not assume σ to have finite order.

If $\sigma \in \Sigma$, we define the σ-twisted conjugation C^σ so that if $h, \gamma \in G$,

$$C^\sigma(h)\gamma = h\gamma\sigma(h^{-1}). \tag{1.2}$$

Let $Z_\sigma(\gamma) \subset G$ be the σ-twisted centralizer of $\gamma \in G$. Then σ-twisted conjugacy class of γ in G can be identified with $Z_\sigma(\gamma)\backslash G$. The twisted orbital integral, defined as a certain integral on $Z_\sigma(\gamma)\backslash G$, has been vastly studied in cyclic base change theory (cf. [1, 4, 15, 32], etc).

Due to the possible nontrivial large center of G, the Lie group $G \rtimes \Sigma$, even G^σ, may fail to be reductive. A typical example is $\mathbb{R}^m \rtimes \mathrm{O}(m)$. In Subsection 2.4, we explain the key point that the above groups do not displace very far from a reductive one. In particular, if $\gamma \in G$ is such that $\gamma\sigma$ is semisimple as an isometry of X, we establish, via a geometric argument, a decomposition theorem for $Z_\sigma(\gamma)$. Then we show that $X(\gamma\sigma)$ is a symmetric space associated with $Z^0_\sigma(\gamma)$, the identity component of $Z_\sigma(\gamma)$. This way, in Subsection 3.2, we give a geometric interpretation for the twisted orbital integrals, as an extension of [9, Definition 4.2.2].

We now assume (E, ρ^E) to be a unitary representation of K^σ. Then the action of G^σ on X lifts to F. The operator $\mathcal{L}^X$ commutes with G^σ. For $t > 0$, let $\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)]$ denote the σ-twisted orbital integral of the kernel of $\exp(-t\mathcal{L}^X)$ associated with γ.

1.3. Statement of the main results

If $\gamma\sigma$ is semisimple, after conjugation, we may and we will assume that $\gamma = \mathrm{e}^a\, k^{-1}$ with $a \in \mathfrak{p}, k \in K$ and $\mathrm{Ad}(k)a = \sigma a$. Then θ acts on $Z_\sigma(\gamma)$. Let

$\mathfrak{z}_\sigma(\gamma) \subset \mathfrak{g}$ be the Lie algebra of $Z_\sigma(\gamma)$, and let $\mathfrak{k}_\sigma(\gamma)$ be the Lie algebra of $K_\sigma(\gamma) = Z_\sigma(\gamma) \cap K$. As in (1.1), we have the Cartan decomposition

$$\mathfrak{z}_\sigma(\gamma) = \mathfrak{p}_\sigma(\gamma) \oplus \mathfrak{k}_\sigma(\gamma). \tag{1.3}$$

Put $p = \dim \mathfrak{p}_\sigma(\gamma)$, $q = \dim \mathfrak{k}_\sigma(\gamma)$.

The analytic function $J_{\gamma\sigma}(Y_0^{\mathfrak{k}})$ in $Y_0^{\mathfrak{k}} \in \mathfrak{k}_\sigma(\gamma)$ will be defined in Definition 4.1 by an explicit formula. The main result of this paper is as follows.

Theorem 1.1. — *For $t > 0$, the following identity holds:*

$$\operatorname{Tr}^{[\gamma\sigma]}\left[\exp(-t\mathcal{L}^X)\right] = \frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}} \tag{1.4}$$
$$\cdot \int_{\mathfrak{k}_\sigma(\gamma)} J_{\gamma\sigma}(Y_0^{\mathfrak{k}}) \operatorname{Tr}^E\left[\rho^E(k^{-1}\sigma)\exp(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}}))\right] \exp(-|Y_0^{\mathfrak{k}}|^2/2t)\frac{\mathrm{d}Y_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.$$

If $\sigma = \mathrm{Id}_G$, it is just Bismut's formula given in [9, Theorem 6.1.1]. In [9, Sections 8.1 and 10.6], Bismut explained that a formula like (1.4) holds for the cases such as $G = K$ non-connected, and $G = \mathbb{R}^m$, $\sigma \in \mathrm{O}(m)$. Our theorem here confirms his observation in a more general setting. We will restate the above theorem in Subsection 4.2, and the proof will be given in Section 6, which is partly derived from [9, Chapter 9]. In Subsection 4.5, we exploit the formula (1.4) in the context of cyclic base change theory over $\mathbb{R}$, so that we only need elementary computation from linear algebra to establish some nontrivial identities.

Let $\mathfrak{p}_\sigma^\perp(\gamma) \subset \mathfrak{p}$ be the orthogonal space of $\mathfrak{p}_\sigma(\gamma)$ in $\mathfrak{p}$ with respect to B. Let $P_\sigma^\perp(\gamma) \subset X$ be the image of $\mathfrak{p}_\sigma^\perp(\gamma)$ by the map $f \to p\,\mathrm{e}^f$. Put

$$\Delta_X^{\gamma\sigma} = \{(x, \gamma\sigma(x)) \,:\, x \in P_\sigma^\perp(\gamma)\} \subset X \times X. \tag{1.5}$$

Let $(a, \mathfrak{k}_\sigma(\gamma))$ denote the affine subspace of $\mathfrak{z}_\sigma(\gamma) = \mathfrak{p}_\sigma(\gamma) \oplus \mathfrak{k}_\sigma(\gamma)$. Set

$$H_\sigma^\gamma = \{0\} \times \big(a, \mathfrak{k}_\sigma(\gamma)\big) \subset \mathfrak{z}_\sigma(\gamma) \times \mathfrak{z}_\sigma(\gamma). \tag{1.6}$$

Let $\Delta^{\mathfrak{z}_\sigma(\gamma)}$ denote the standard Laplacian on $\mathfrak{z}_\sigma(\gamma)$.

In Subsection 4.3, using Theorem 1.1, we get an extension of [9, Theorem 6.3.2] for the twisted orbital integrals for wave operators.

Theorem 1.2. — *We have the identity of even distributions on $\mathbb{R}$ (defined in Subsection 4.3) supported on $\{s \in \mathbb{R} : |s| \geqslant \sqrt{2}|a|\}$ with singular support included in $\pm\sqrt{2}|a|$,*

$$\int_{\Delta_X^{\gamma\sigma}} \operatorname{Tr}^F\left[\gamma\sigma\cos\left(s\sqrt{\mathcal{L}^X}\right)\right] \tag{1.7}$$
$$= \int_{H_\sigma^\gamma} \operatorname{Tr}^E\left[\cos\left(s\sqrt{-\Delta^{\mathfrak{z}_\sigma(\gamma)}/2}\right) J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\rho^E(k^{-1}\sigma)\exp\left(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}})\right)\right].$$

1.4. Hypoelliptic Laplacian on symmetric spaces

Let us briefly recall the theory of hypoelliptic Laplacian developed by Bismut in [9]. We also refer to [38] for an introduction to this theory.

Put $N = G\times_K \mathfrak{k}$. Then $TX\oplus N$ is canonically trivial on X. Let $\widehat{\pi} : \widehat{\mathcal{X}} \to X$ be the total space of $TX\oplus N$, so that $\widehat{\mathcal{X}} = X\times \mathfrak{g}$. The hypoelliptic Laplacian is defined as a family of hypoelliptic differential operators $\{\mathcal{L}^X_b\}_{b>0}$ acting on $C^\infty(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F))$.

Let $\Delta^{TX\oplus N}$ be the standard Laplace along the fibre $TX \oplus N$. Then $\mathcal{L}^X_b$ is given as follows [9, Section 2.13],

$$(1.8)\quad \mathcal{L}^X_b = \frac{1}{2}\left|[Y^N, Y^{TX}]\right|^2 + \frac{1}{2b^2}\left(-\Delta^{TX\oplus N} + |Y|^2 - m - n\right) + \frac{N^{\Lambda^\bullet(T^*X\oplus N^*)}}{b^2} + \frac{1}{b}\Big(\nabla^{C^\infty(TX\oplus N,\widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F))}_{Y^{TX}} + \widehat{c}\big(\mathrm{ad}(Y^{TX})\big) - c\big(\mathrm{ad}(Y^{TX}) + \mathrm{i}\theta\,\mathrm{ad}(Y^N)\big) - \mathrm{i}\rho^E(Y^N)\Big).$$

The structure of $\mathcal{L}^X_b$ is close to the structure of the hypoelliptic Laplacian studied by Bismut [7, 8], and by Bismut–Lebeau [10].

In [9], the proper functional analytic machinery was developed in order to obtain the analytic properties of the resolvent and of the heat kernel of $\mathcal{L}^X_b$. Let $\exp(-t\mathcal{L}^X_b)$ be the heat operator associated with $\mathcal{L}^X_b$. In [9, Chapters 11, 14], Bismut proved that there is a smooth kernel $q^X_{b,t}$ associated with $\exp(-t\mathcal{L}^X_b)$, and that as $b \to 0$, $q^X_{b,t}$ converges in the proper sense to the kernel of $\exp(-t\mathcal{L}^X)$.

In (1.8), the term $\nabla^{C^\infty(TX\oplus N,\widehat{\pi}^*(\Lambda^\bullet(T^*X\oplus N^*)\otimes F))}_{Y^{TX}}$ represents the left action of the generator of the geodesic flow. If we forget the first quartic term in the right-hand side of (1.8), then after rescaling, as $b \to +\infty$, $\mathcal{L}^X_b$ converges in a naïve sense to the generator of the geodesic flow. More precisely, the diffusion associated with the scalar part of $\mathcal{L}^X_b$ tends to propagate along the geodesic flow. In [9, Chapters 12, 15], Bismut established the uniform estimates on $q^X_{b,t}$ for b large, from which he gave a quantitative estimate on how much this diffusion differs from the geodesic flow.

In Subsection 3.3, we also define the σ-twisted orbital integral for the (hypoelliptic) heat kernel of $\mathcal{L}^X_b$. Then in Theorem 6.1, we establish an identity which says that, for $b > 0$, $t > 0$,

$$(1.9)\qquad \mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)] = \mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_b)].$$

Theorem 1.1 is obtained by evaluating the right-hand side of (1.9) as $b \to +\infty$. As we will explain in Subsection 6.3, this evaluation can be localized near $X(\gamma\sigma)$, more precisely, the $\gamma\sigma$-periodic points of the geodesic flow on $\widehat{\mathcal{X}}$. Then, using methods of local index theory, we can explicitly work out its limit as $b \to +\infty$ and obtain (1.4).

1.5. Local equivariant index theorems

Let Γ be a cocompact torsion-free discrete subgroup of G such that $\sigma(\Gamma) = \Gamma$. Then $Z = \Gamma\backslash X$ is a compact smooth manifold equipped with an isometric action of Σ^σ. The vector bundle F descends to one on Z, so that the action of Σ^σ on Z lifts to $F \to Z$. Moreover, $\mathcal{L}^X$ descends to a Bochner-like Laplacian $\mathcal{L}^Z$ on Z, whose heat operators are trace class.

The twisted trace formula shows,

$$\begin{aligned}(1.10)\quad \operatorname{Tr}\left[\sigma^Z \exp(-t\mathcal{L}^Z)\right] \\ = \sum_{\underline{[\gamma]}_\sigma \in [\Gamma]_\sigma} \operatorname{Vol}\left(\Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma)\right) \operatorname{Tr}^{[\gamma\sigma]}\left[\exp(-t\mathcal{L}^X)\right],\end{aligned}$$

where $[\Gamma]_\sigma$ is the set of σ-twisted conjugacy classes in Γ.

Under the geometric setting in Section 7, the operator $\mathcal{L}^Z$ can be replaced by the Laplacian for spinors, or the Hodge Laplacian for a Hermitian flat vector bundle (F is defined by a G^σ-representation (E, ρ^E)). Then, combining (1.4) with (1.10), we get a formula for the σ-equivariant heat trace, from which we can evaluate the equivariant Dirac index or σ-equivariant real analytic torsion.

For the example of equivariant Euler characteristic number $\chi_\sigma(Z, F)$ with a flat vector bundle F ($\mathcal{L}^Z$ is the Hodge Laplacian $\mathbf{D}^{Z,F,2}$ up to a parallel endomorphism of F), we will show that all the term $\operatorname{Tr}^{[\gamma\sigma]}[\exp(-t\mathbf{D}^{X,F,2})]$ in (1.10) vanish except for the elliptic class $\underline{[\gamma]}_\sigma$ (i.e., $\gamma\sigma$ has fixed points in X). Let $\underline{E}_\sigma \subset [\Gamma]_\sigma$ denote the finite set of elliptic classes, and let ${}^\sigma Z \subset Z$ denote the fixed point set of isometry σ. Then, in Subsection 2.6, we get

$$(1.11)\qquad {}^\sigma Z = \bigcup_{\underline{[\gamma]}_\sigma \in \underline{E}_\sigma}^{\text{disjoint}} \Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma),$$

where $X(\gamma\sigma)$ is just the fixed point set of $\gamma\sigma$ in X.

In Subsections 7.2 and 7.4, we exhibit how to proceed a further evaluation on the integral in the right-hand side of (1.4) by analyzing the representation (E, ρ^E). In particular, if $\underline{[\gamma]}_\sigma \in \underline{E}_\sigma$,

$$\text{(1.12)} \quad \mathrm{Tr}^{[\gamma\sigma]}\left[\exp(-t\mathbf{D}^{X,F,2})\right] = \left[e\left(TX(\gamma\sigma), \nabla^{TX(\gamma\sigma)}\right)\right]^{\max} \mathrm{Tr}^E\left[\rho^E(\gamma\sigma)\right],$$

where $e\left(TX(\gamma\sigma), \nabla^{TX(\gamma\sigma)}\right)$ denotes the Euler form of $X(\gamma\sigma)$, hence identified locally with the Euler form of ${}^\sigma Z$. Finally, we assembly together the above computations, we get

$$\text{(1.13)} \qquad \chi_\sigma(Z, F) = \sum_{\underline{[\gamma]}_\sigma \in \underline{E}_\sigma} \chi\Big(\Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma)\Big) \mathrm{Tr}^E\left[\rho^E(\gamma\sigma)\right],$$

where $\chi(\cdots)$ denotes the corresponding Euler characteristic number. This is clearly a specialization of the local equivariant index theorem (cf. [6, Chapter 6]) for the locally symmetric space Z.

In the last subsection, we introduce the σ-twisted L_2-torsion $\mathcal{T}_{\sigma,L_2}(Z, F)$, as an extension of the definition in [4]. We explain briefly that $\mathcal{T}_{\sigma,L_2}(Z, F)$ plays a similar role as the ordinary L_2-torsions ([37, 39]), but associated with the fixed point set ${}^\sigma Z$.

1.6. The organization of the paper

This paper is organized as follows. In Section 2, we introduce the real reductive Lie group G and the twist σ, and we explain the associated geometric structure for $X(\gamma\sigma)$ when $\gamma\sigma$ is semisimple.

In Section 3, we establish the geometric formulation for the σ-twisted orbital integrals associated with γ.

In Section 4, we restate Theorem 1.1 as Theorem 4.6, and we give a vanishing theorem by classifying the representations of K^σ. We also explain our formula for the examples from cyclic base change theory.

In Section 5, we recall the construction of the hypoelliptic Laplacian $\mathcal{L}_b^X$ of Bismut and the properties of its heat kernel [9].

In Section 6, we prove Theorem 4.6.

Finally, in Section 7, we show the compatibility of our formula (1.4) with the local equivariant index theorems for compact locally symmetric spaces. In the last part, we discuss briefly the twisted L_2-torsion introduced in [4].

This paper is mainly the first part of the author's thesis [34], and the main results were announced in [35].

In the sequel, if V is a real vector space and if E is a complex vector space, we will denote by $V \otimes E$ the complex vector space $V \otimes_{\mathbb{R}} E$. We use the same convention for the tensor product of vector bundles.

If H is a Lie group, let H^0 denote the connected component of identity.

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2. The symmetric space $X = G/K$ and semisimple isometries

In this section, we consider a connected real reductive Lie group G, and let X be the associated symmetric space. We introduce a compact subgroup Σ of $\mathrm{Aut}(G)$ which acts on X isometrically, then for each semisimple element $\gamma\sigma := (\gamma, \sigma) \in G \rtimes \Sigma$, we construct a symmetric space $X(\gamma\sigma) \subset X$ associated with the σ-twisted centralizer of γ in G. Our results here are direct extensions of the results obtained in [9, Chapter 3]. They are necessary to establish the geometric formulation of the twisted orbital integrals in Section 3.

2.1. Symmetric spaces and homogeneous vector bundles

Let G be a connected real reductive Lie group [29, §7.2] with a Cartan involution θ. Let $K \subset G$ be the fixed point set of θ, which is a connected maximal compact subgroup. Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of G, K respectively. Let $\mathfrak{p} \subset \mathfrak{g}$ be the eigenspace of θ associated with the eigenvalue -1. Then the Cartan decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{2.1}$$

Moreover,

$$[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p},\quad [\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k},\quad [\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}. \tag{2.2}$$

Put $m=\dim\mathfrak{p}, n=\dim\mathfrak{k}$. Then $\dim\mathfrak{g}=m+n$

Let B be a nondegenerate bilinear symmetric form on $\mathfrak{g}$ which is positive-definite on $\mathfrak{p}$ and negative-definite on $\mathfrak{k}$. We also assume that B is invariant under the action of θ and the adjoint action of G. Let $\langle\cdot\,,\cdot\rangle$ be the scalar product on $\mathfrak{g}$ defined by $-B(\cdot,\theta\cdot)$. Then the splitting (2.1) is orthogonal with respect to B and $\langle\cdot\,,\cdot\rangle$.

For $g,g'\in G$, put

$$C(g)g'=gg'g^{-1}\in G. \tag{2.3}$$

Let $\mathrm{Ad}(\cdot)$, $\mathrm{ad}(\cdot)$ denote respectively the adjoint actions of G, $\mathfrak{g}$ on $\mathfrak{g}$. We also use $\mathrm{Ad}(g)$ abusively to denote the conjugation $C(g)$ on G.

Let $\omega^{\mathfrak{g}}=\omega^{\mathfrak{p}}+\omega^{\mathfrak{k}}$ be the canonical left-invariant 1-form on G with values in $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k}$. Then by (2.1), (2.2), we get

$$\mathrm{d}\omega^{\mathfrak{p}}=-[\omega^{\mathfrak{k}},\omega^{\mathfrak{p}}],\quad \mathrm{d}\omega^{\mathfrak{k}}=-\frac{1}{2}[\omega^{\mathfrak{k}},\omega^{\mathfrak{k}}]-\frac{1}{2}[\omega^{\mathfrak{p}},\omega^{\mathfrak{p}}]. \tag{2.4}$$

Let $X=G/K$ be the associated symmetric space. The projection $p: G\to G/K$ defines a K-principal bundle on X, then the splitting (2.1) gives it a connection with the connection form $\omega^{\mathfrak{k}}$. Let Ω be the associated curvature, then by (2.4),

$$\Omega=-\frac{1}{2}[\omega^{\mathfrak{p}},\omega^{\mathfrak{p}}]\in\Lambda^2(\mathfrak{p}^*)\otimes\mathfrak{k}. \tag{2.5}$$

If (E,ρ^E,h^E) is a finite dimensional orthogonal (resp. unitary) representation of K, then $(F=G\times_K E,h^F)$ is a Euclidean (resp. Hermitian) vector bundle on X. The connection form $\omega^{\mathfrak{k}}$ induces a Euclidean (resp. Hermitian) connection ∇^F on F. The actions of G and θ on X lift to F.

Note that K acts on $\mathfrak{p}$ via adjoint action. Then we have the identification

$$TX=G\times_K\mathfrak{p}. \tag{2.6}$$

Moreover, $B|_{\mathfrak{p}}$ induces a Riemannian metric g^{TX} on TX. Then G and θ act on X isometrically. Let $d(\cdot\,,\cdot)$ denote the Riemannian distance on X. By (2.4), (2.6), $\omega^{\mathfrak{k}}$ induces the Levi–Civita connection ∇^{TX} on (TX,g^{TX}). Let R^{TX} denote its curvature. Then X has nonpositive sectional curvature. If $x_0=p1\in X$, the exponential map $\exp_{x_0}:\mathfrak{p}\to X$ given by $Y^{\mathfrak{p}}\in\mathfrak{p}\to\exp_{x_0}(Y^{\mathfrak{p}})=\exp(Y^{\mathfrak{p}})\cdot x_0$ is a diffeomorphism between $\mathfrak{p}$ and X.

Put

$$N=G\times_K\mathfrak{k}. \tag{2.7}$$

Let ∇^N be the connection on N associated with $\omega^{\mathfrak{k}}$. By (2.6), eqrefeq:1.1.6n,

$$TX \oplus N = G \times_K \mathfrak{g}. \tag{2.8}$$

Let $\nabla^{TX\oplus N}$ be the connection on $TX \oplus N$ associated with $\omega^{\mathfrak{k}}$, equivalently, $\nabla^{TX\oplus N} = \nabla^{TX} \oplus \nabla^N$.

In the sequel, let $\pi : \mathcal{X} \to X$ be the total space of TX to X, and let $\widehat{\pi} : \widehat{\mathcal{X}} \to X$ be the total space of $TX \oplus N$ to X. The map $(g, a) \in G \times_K \mathfrak{g} \to (pg, \mathrm{Ad}(g)a) \in X \times \mathfrak{g}$ identifies $TX \oplus N$ with the trivial vector bundle $\mathfrak{g}$ over X. Then

$$\widehat{\mathcal{X}} \simeq X \times \mathfrak{g}. \tag{2.9}$$

We now go back to the Hermitian vector bundle F on X associated with a unitary representation (E, ρ^E) of K. Let $C^\infty(G, E)$ denote the set of smooth functions on G valued in E. The right multiplication of K on G induces a dot-action of K on $C^\infty(G, E)$, such that for $k \in K$, $s \in C^\infty(G, E)$,

$$(k.s)(g) = \rho^E(k)s(gk). \tag{2.10}$$

Let $C^\infty_K(G, E)$ be the subspace of $C^\infty(G, E)$ of the sections fixed by K-dot-action. Let $C^\infty(X, F)$ be the vector space of the smooth sections of F over X. Then

$$C^\infty(X, F) = C^\infty_K(G, E). \tag{2.11}$$

The left action of G on itself induces an equivariant action of G on $C^\infty(X, F)$ such that if $s \in C^\infty_K(G, E)$, if $g, h \in G$, then

$$(hs)(g) = s(h^{-1}g). \tag{2.12}$$

Moreover, ∇^F is G-invariant.

2.2. Semisimple isometries of X

Let $\mathrm{Isom}(X)$ be the Lie group of isometries of (X, g^{TX}). Then we have an obvious group homomorphism $G \to \mathrm{Isom}(X)$.

Definition 2.1. — *If $\phi \in \mathrm{Isom}(X)$, the displacement function d_ϕ associated with ϕ is the function on X defined as*

$$d_\phi(x) = d(x, \phi x)\ ,\ x \in X. \tag{2.13}$$

Put $m_\phi = \inf_{x\in X} d_\phi(x) \in \mathbb{R}_{\geqslant 0}$.

Since X has nonpositive sectional curvature, by [21, Chapter 1, Example 1.6.6], d^2_ϕ is a smooth convex function on X.

Definition 2.2. — *We say that* $\phi \in \mathrm{Isom}(X)$ *is semisimple if* $d_\phi(x)$ *reaches its infimum* m_ϕ *in* X. *A semisimple isometry* ϕ *is called elliptic if it has fixed points in* X, *i.e.* $m_\phi = 0$. *If* ϕ *is semisimple, put* $X(\phi) = \{x \in X \mid d_\phi(x) = m_\phi\}$.

2.3. A compact subgroup of $\mathrm{Aut}(G)$

Let $\mathrm{Aut}(G)$ be the Lie group of automorphism of G [23, Theorem 2].

Definition 2.3. — *The semidirect product of* G *and* $\mathrm{Aut}(G)$ *is defined as*

$$G \rtimes \mathrm{Aut}(G) := \{(g, \phi) \ : \ g \in G, \phi \in \mathrm{Aut}(G)\}, \tag{2.14}$$

with the group multiplication:

$$(g_1, \phi_1) \cdot (g_2, \phi_2) = \big(g_1\phi_1(g_2), \phi_1\phi_2\big). \tag{2.15}$$

The unit element is $(1, \mathrm{Id}_G)$. *Also* $(g, \phi)^{-1} = (\phi^{-1}(g^{-1}), \phi^{-1})$.

We will often write $g\phi$ instead of (g, ϕ) for an element in $G \rtimes \mathrm{Aut}(G)$.

Definition 2.4. — *Put*

$$\Sigma := \{\phi \in \mathrm{Aut}(G) \ : \ \phi\theta = \theta\phi, \ \phi \text{ preserves the bilinear form } B\}. \tag{2.16}$$

Then Σ is a compact Lie subgroup of $\mathrm{Aut}(G)$. Let $\mathfrak{e}$ be its Lie algebra. The action of Σ on $\mathfrak{g}$ preserves the splitting (2.1) and the scalar product of $\mathfrak{g}$. Note that Σ contains all the inner automorphisms defined by elements in K.

Set

$$\widetilde{G} = G \rtimes \Sigma, \ \widetilde{K} = K \rtimes \Sigma. \tag{2.17}$$

They are closed subgroups of $G \rtimes \mathrm{Aut}(G)$. Let $\widetilde{\mathfrak{g}}$, $\widetilde{\mathfrak{k}}$ denote their Lie algebras respectively. As vector spaces, we have $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{e}$, $\widetilde{\mathfrak{k}} = \mathfrak{k} \oplus \mathfrak{e}$. Then we have the Cartan splitting of $\widetilde{\mathfrak{g}}$ (associated with the conjugation of θ),

$$\widetilde{\mathfrak{g}} = \mathfrak{p} \oplus \widetilde{\mathfrak{k}}. \tag{2.18}$$

Moreover, we also have the global Cartan decomposition for $\mathfrak{p} \times \widetilde{K} \simeq \widetilde{G}$, where the diffeomorphism is given by $(f, \widetilde{k}) \mapsto \exp(f)\widetilde{k} \in \widetilde{G}$.

Remark 2.5. — The group $\widetilde{G}$ is not necessarily reductive. An example is the case $\mathbb{R}^m$. In this case $\widetilde{G} = \mathbb{R}^m \rtimes \mathrm{O}(m)$ and the corresponding Lie algebra is $\widetilde{\mathfrak{g}} = \mathbb{R}^m \oplus \mathfrak{so}(m)$ with a twisted Lie bracket.

Given $\sigma \in \Sigma$, the map $g \in G \to \sigma(g) \in G$ descends to a diffeomorphism of X: $x \in X \to \sigma(x) \in X$. By (2.6), (2.16), the derivative of σ is given by $(g, f) \to (\sigma(g), \sigma(f))$ with $g \in G, f \in \mathfrak{p}$. Then $\widetilde{G}$ acts on X isometrically. Then

$$X = \widetilde{G}/\widetilde{K}. \tag{2.19}$$

Fix $\sigma \in \Sigma$, let Σ^σ be the closure of the subgroup of Σ generated by σ. Set

$$G^\sigma = G \rtimes \Sigma^\sigma,\ K^\sigma = K \rtimes \Sigma^\sigma. \tag{2.20}$$

Similarly to (2.19),

$$X = G^\sigma/K^\sigma. \tag{2.21}$$

Moreover, we have

$$TX = \widetilde{G} \times_{\widetilde{K}} \mathfrak{p} = G^\sigma \times_{K^\sigma} \mathfrak{p}. \tag{2.22}$$

In the sequel, p denotes both the projections $\widetilde{G} \to X$ and $G^\sigma \to X$.

If the representation $\rho^E : K \to \mathrm{Aut}(E)$ extends to a representation $\rho^E : K^\sigma \to \mathrm{Aut}(E)$, then we have the identification of vector bundles over X,

$$F = G^\sigma \times_{K^\sigma} E. \tag{2.23}$$

The question on such extensions is studied in Subsection 4.4. In this case, the equivariant action of σ on F is represented by $\sigma(g, f) \to (\sigma(g), \rho^E(\sigma)f)$. Moreover, as in (2.11), we have

$$C^\infty(X, F) = C^\infty_{K^\sigma}(G^\sigma, E). \tag{2.24}$$

Then G^σ acts on $C^\infty(X, F)$. Also ∇^F is invariant under the action of G^σ.

2.4. The decomposition of semisimple elements in $\widetilde{G}$

Definition 2.6. — *An element of $\widetilde{G}$ is semisimple (resp. elliptic) if its isometric action on X is semisimple (resp. elliptic).*

If $\gamma \in \widetilde{G}$, let $\widetilde{Z}(\gamma)$ be the centralizer of γ in $\widetilde{G}$. Then d_γ is $\widetilde{Z}(\gamma)$-invariant. Recall that if γ is semisimple, $X(\gamma)$ is the minimizing set of d_γ.

Using instead the identification (2.19), and by the geometric properties of (X, g^{TX}), the same arguments in the proof of [9, Theorem 3.1.2] give the following criterion on the set $X(\gamma)$.

THEOREM 2.7. — *Assume that $\gamma \in \widetilde{G}$ is semisimple. If $g \in \widetilde{G}$, $x = pg \in X$, then $x \in X(\gamma)$ if and only if there exist $a \in \mathfrak{p}$, $k \in \widetilde{K}$ such that $\mathrm{Ad}(k)a = a$ and $\gamma = C(g)(\mathrm{e}^a\, k^{-1})$. If $g_t = g\,\mathrm{e}^{ta}$, then $t \in [0,1] \to x_t = pg_t$ is the unique geodesic connecting x and γx. Moreover, $m_\gamma = |a|$, and $k \in \widetilde{K}$ is the parallel transport along the above geodesic.*

By Theorem 2.7, $\gamma \in \widetilde{G}$ is elliptic if and only if it is conjugate in $\widetilde{G}$ to an element of $\widetilde{K}$. An element $\gamma \in \widetilde{G}$ is said to be hyperbolic if it is conjugate in $\widetilde{G}$ to $\mathrm{e}^a, a \in \mathfrak{p}$, which is always semisimple. Moreover, a hyperbolic element always lies in G, and can be conjugate to $\exp(\mathfrak{p})$ by an element in G.

If $a \in \mathfrak{g}$, let $Z(a) \subset G$, $\widetilde{Z}(a) \subset \widetilde{G}$ be the stabilizers of a, and let $\mathfrak{z}(a) \subset \mathfrak{g}$, $\widetilde{\mathfrak{z}}(a) \subset \widetilde{\mathfrak{g}}$ be their Lie algebras. If $a \in \mathfrak{p}$, by the same arguments as in the proof to [9, Proposition 3.2.8], we have

$$Z(a) = Z(\mathrm{e}^a),\ \widetilde{Z}(a) = \widetilde{Z}(\mathrm{e}^a). \tag{2.25}$$

Also we have,

$$\widetilde{\mathfrak{z}}(a) = \{f \in \widetilde{\mathfrak{g}}\ :\ [f,a] = 0\},\ \mathfrak{z}(a) = \widetilde{\mathfrak{z}}(a) \cap \mathfrak{g}, \tag{2.26}$$

The group $\widetilde{G}$ may fail to be a reductive Lie group, but it is not far from it (twisted by a compact group), so that $\widetilde{G}$ still has the properties of a connected reductive Lie group discussed in [21, Theorem 2.19.23] and [9, Subsection 3.1]. For the sake of completeness, we include proofs to these properties.

PROPOSITION 2.8. — *Assume that $\gamma \in \widetilde{G}$ is such that*

$$\gamma = \mathrm{e}^a\, k^{-1},\ a \in \mathfrak{p},\ k \in \widetilde{K},\ \mathrm{Ad}(k)a = a. \tag{2.27}$$

Then we have

$$\widetilde{Z}(\gamma) = \widetilde{Z}(\mathrm{e}^a) \cap \widetilde{Z}(k^{-1}). \tag{2.28}$$

Proof. — By Theorem 2.7, γ is semisimple, and $x_0 = p1 \in X(\gamma)$. We only need to prove that $\widetilde{Z}(\mathrm{e}^a) \cap \widetilde{Z}(k^{-1}) \supset \widetilde{Z}(\gamma)$. We will adapt the arguments of [9, Theorem 3.2.6 and Proposition 3.2.8].

Take $h \in \widetilde{Z}(\gamma)$. Then there exists unique $f \in \mathfrak{p}$ and $k' \in \widetilde{K}$ such that $h = \mathrm{e}^f\, k'$. Then $\gamma x_0, hx_0 = p\,\mathrm{e}^f, \gamma h x_0 \in X(\gamma)$.

Let $y_s = p\,\mathrm{e}^{sa}, s \in [0,1]$ be the unique geodesic in X joining x_0 and γx_0. Let $x_t = p\,\mathrm{e}^{tf}, t \in [0,1]$ be the unique geodesic connecting x_0 and hx_0. Since $X(\gamma)$ is geodesically convex, then the paths $y_\cdot, x_\cdot$ lie in $X(\gamma)$. Also we have two other geodesics $\gamma x_\cdot, h y_\cdot$ in $X(\gamma)$. These four geodesics form a geodesic rectangle in $X(\gamma)$ with the vertexes $x_0, \gamma x_0, hx_0, \gamma h x_0 = h\gamma x_0$.

Let $c_t(s), 0 \leqslant s \leqslant 1$ be the geodesic connecting x_t and γx_t for all t. If $t \in [0,1]$, let $E_f(t)$ be the energy function associated with $c_t(\cdot)$, i.e.,

$$E_f(t) = \frac{1}{2} d^2_\gamma(x_t). \tag{2.29}$$

In particular, $E_f(t)$ is a constant function in t, so that

$$E''_f(0) = 0. \tag{2.30}$$

Put $J_s = \frac{\partial}{\partial t}|_{t=0} c_t(s)$ the Jacobi field along $c_0(s)$. In the trivialization by parallel transport,

$$\begin{aligned} &\ddot{J}_s - \mathrm{ad}^2(a) J_s = 0, \\ &J_0 = f, \quad J_1 = \mathrm{Ad}(k^{-1}) f, \end{aligned} \tag{2.31}$$

where $\dot{J}, \ddot{J}$ are taken with respect to the Levi–Civita connection along $y_\cdot$.

We also have

$$E''_f(0) = \int_0^1 \left(|\dot{J}_s|^2 + \left|[a, J_s]\right|^2 \right) \mathrm{d}s. \tag{2.32}$$

By (2.30), (2.31), (2.32), we get

$$f \in \mathfrak{z}(a) \cap \mathfrak{p}, \quad \mathrm{Ad}(k) f = f. \tag{2.33}$$

Applying (2.33) to $h = \mathrm{e}^f k'$, $h\gamma = \gamma h$, we obtain

$$\mathrm{e}^{\mathrm{Ad}(k')a} k' k^{-1} = \mathrm{e}^a k^{-1} k'. \tag{2.34}$$

Using the uniqueness of global Cartan decomposition of G, we get

$$\mathrm{Ad}(k') a = a, \; k' k^{-1} = k^{-1} k'. \tag{2.35}$$

By (2.33), (2.35), we get $h \in \widetilde{Z}(\mathrm{e}^a) \cap \widetilde{Z}(k^{-1})$. This completes our proof. □

In general, if $\gamma \in \widetilde{G}$ is semisimple, then by Theorem 2.7, there exist $g \in \widetilde{G}$, $a \in \mathfrak{p}$, $k \in \widetilde{K}$ such that

$$\gamma = g\, \mathrm{e}^a\, k^{-1} g^{-1}, \; \mathrm{Ad}(k) a = a. \tag{2.36}$$

Put

$$\gamma_h = g\, \mathrm{e}^a\, g^{-1}, \quad \gamma_e = g k^{-1} g^{-1}. \tag{2.37}$$

The element γ_h (resp. γ_e) is called the hyperbolic (resp. elliptic) part of γ. Then $\gamma = \gamma_h \gamma_e = \gamma_e \gamma_h$. By Proposition 2.8,

$$\widetilde{Z}(\gamma) = \widetilde{Z}(\gamma_e) \cap \widetilde{Z}(\gamma_h). \tag{2.38}$$

Lemma 2.9. — *If $\gamma \in \widetilde{G}$ is semisimple, then the decomposition of γ as the commuting product of a hyperbolic element and an elliptic element in $\widetilde{G}$ is unique.*

Proof. — It is enough to prove our lemma for γ given in (2.27), where we have

$$\gamma_h = \mathrm{e}^a \in G,\ \gamma_e = k^{-1} \in \widetilde{K}. \tag{2.39}$$

Now suppose that $\gamma'_h \in G, \gamma'_e \in \widetilde{G}$ are respectively hyperbolic, elliptic elements such that

$$\gamma = \gamma'_h \gamma'_e = \gamma'_e \gamma'_h. \tag{2.40}$$

Then we only need to prove that

$$\gamma'_h = \gamma_h,\ \gamma'_e = \gamma_e. \tag{2.41}$$

Note that the conjugation of $\widetilde{G}$ preserves G, then $\gamma'_h \in G$. Set

$$H = \ker(\mathrm{Ad} : \widetilde{G} \to \mathrm{Aut}(\mathfrak{g})). \tag{2.42}$$

Then $H \cap G$ is just the center of G.

Then the uniqueness of the Jordan decomposition of $\mathrm{Ad}(\gamma)$ implies

$$\mathrm{Ad}(\gamma'_h) = \mathrm{Ad}(\gamma_h),\ \mathrm{Ad}(\gamma'_e) = \mathrm{Ad}(\gamma_e) \in \mathrm{Aut}(\mathfrak{g}). \tag{2.43}$$

This implies that there exists $h \in H \cap G \cap \widetilde{Z}(\gamma'_e) \cap \widetilde{Z}(\gamma_e)$ such that

$$\gamma'_h = h\gamma_h,\ \gamma'_e = h^{-1}\gamma_e. \tag{2.44}$$

Write $h = \mathrm{e}^f k'' \in G$ with $f \in \mathfrak{p}$, $k'' \in K$. Then by Theorem 2.7 and the assumption that γ'_e is elliptic, we get $f = 0$, so that $h \in K$ and $\gamma'_e \in K$.

Since γ'_h is hyperbolic, then there exist $g' \in G$, $a' \in \mathfrak{p}$ such that $\gamma'_h = g\,\mathrm{e}^{a'} g^{-1}$. Then we rewrite the first identity of (2.44) as follows

$$g\,\mathrm{e}^{a'} g^{-1} = \mathrm{e}^a\, h \in G,\ \mathrm{Ad}(h)a = a. \tag{2.45}$$

Using the uniqueness of the elliptic part of a semisimple element in G (cf. [21, Theorem 2.19.23]), we get $h = 1$, which implies exactly (2.41). This completes the proof of our lemma. □

2.5. The minimizing set

Take $\gamma \in G$, $\sigma \in \Sigma$ such that $\gamma\sigma \in \widetilde{G}$ is semisimple. For $g \in G$, we have

$$C(g)(\gamma\sigma) = g\gamma\sigma(g^{-1})\sigma \in G^\sigma. \tag{2.46}$$

Let $C^\sigma : G \to G$ be such that if $g, h \in G$,

$$C^\sigma(g)h = gh\sigma(g^{-1}) \in G. \tag{2.47}$$

If $g \in G$, $C^\sigma(g)$ acts on the left on G, and moreover, $C^\sigma(g)C^\sigma(g') = C^\sigma(gg')$.

Definition 2.10. — *If $\gamma \in G$, let $Z_\sigma(\gamma) \subset G$ be the stabilizer of γ under the action of G by C^σ, which is also called the σ-twisted centralizer of γ in G. Then*

$$Z_\sigma(\gamma) = G \cap \widetilde{Z}(\gamma\sigma). \tag{2.48}$$

The orbit of γ under this action is called σ-twisted conjugacy class of γ in G.

Fix $g \in G$ such that $x = pg \in X(\gamma\sigma)$. By Theorem 2.7, there exists $a \in \mathfrak{p}$, $k \in K$ such that

$$\mathrm{Ad}(k)a = \sigma a, \ \ \gamma = C^\sigma(g)(\mathrm{e}^a\, k^{-1}). \tag{2.49}$$

We have $X(\gamma\sigma) = gX(\mathrm{e}^a\, k^{-1}\sigma)$. Then it is enough to consider the case

$$\gamma = \mathrm{e}^a\, k^{-1} \in G,\ a \in \mathfrak{p},\ k \in K,\ \mathrm{Ad}(k)a = \sigma a. \tag{2.50}$$

In the sequel, we always take γ as in (2.50).

By (2.38), we have

$$Z_\sigma(\gamma) = Z(\mathrm{e}^a) \cap Z_\sigma(k^{-1}). \tag{2.51}$$

Let $\mathfrak{z}_\sigma(\gamma)$, $\mathfrak{z}_\sigma(k^{-1})$ be the Lie algebras of $Z_\sigma(\gamma)$, $Z_\sigma(k^{-1})$. Then

$$\mathfrak{z}_\sigma(k^{-1}) = \{f \in \mathfrak{g} \ :\ \mathrm{Ad}(k)f = \sigma f\}. \tag{2.52}$$

By (2.25), (2.51), we get

$$\mathfrak{z}_\sigma(\gamma) = \mathfrak{z}(a) \cap \mathfrak{z}_\sigma(k^{-1}). \tag{2.53}$$

Put

$$\mathfrak{p}_\sigma(\gamma) \coloneqq \mathfrak{z}_\sigma(\gamma) \cap \mathfrak{p}, \ \ \mathfrak{k}_\sigma(\gamma) \coloneqq \mathfrak{z}_\sigma(\gamma) \cap \mathfrak{k}. \tag{2.54}$$

Since σ preserves the splitting (2.1), by (2.52), (2.53), we get

$$\mathfrak{z}_\sigma(\gamma) = \mathfrak{p}_\sigma(\gamma) \oplus \mathfrak{k}_\sigma(\gamma). \tag{2.55}$$

Put

$$K_\sigma(\gamma) = Z_\sigma(\gamma) \cap K. \tag{2.56}$$

Then $\mathfrak{k}_\sigma(\gamma)$ is the Lie algebra of $K_\sigma(\gamma)$.

Let $Z^0_\sigma(\gamma)$ denote the identity component of $Z_\sigma(\gamma)$. The following result extends [9, Theorem 3.3.1]. Note that $x_0 = p1 \in X(\gamma\sigma)$.

Theorem 2.11. — *We have*

$$X(\gamma\sigma) = X(\mathrm{e}^a) \cap X(k^{-1}\sigma) \subset X. \tag{2.57}$$

In the coordinate system $(\mathfrak{p}, \exp_{x_0})$, we have

$$X(\mathrm{e}^a) = \mathfrak{p}(a) = \mathfrak{z}(a) \cap \mathfrak{p}, \ \ X(k^{-1}\sigma) = \mathfrak{p}_\sigma(k^{-1}). \tag{2.58}$$

Then

$$X(\gamma\sigma) = \mathfrak{p}_\sigma(\gamma). \tag{2.59}$$

The action of $Z^0_\sigma(\gamma)$ on $X(\gamma\sigma)$ is transitive, and the stabilizer of $x = p1 \in X(\gamma\sigma)$ is given by $Z^0_\sigma(\gamma) \cap K$. Then we have the following identifications,

$$X(\gamma\sigma) \simeq Z_\sigma(\gamma)/K_\sigma(\gamma) \simeq Z^0_\sigma(\gamma)/(Z^0_\sigma(\gamma) \cap K). \tag{2.60}$$

Moreover, $Z^0_\sigma(\gamma) \cap K$ coincides with the identity component $K^0_\sigma(\gamma)$ of $K_\sigma(\gamma)$. The embedding $K_\sigma(\gamma) \to Z_\sigma(\gamma)$ induces the isomorphism of finite groups,

$$K^0_\sigma(\gamma)\backslash K_\sigma(\gamma) \simeq Z^0_\sigma(\gamma)\backslash Z(\gamma). \tag{2.61}$$

Proof. — We only prove (2.57) and (2.58), since other results, as their consequences, follow from the standard arguments on symmetric spaces.

Note that $p : G \to X$ is surjective. For $y \in X(\gamma\sigma)$, there exists $g \in G$ such that $y = pg$. By Theorem 2.7, there exists $a' \in \mathfrak{p}, k' \in K$ such that $\gamma\sigma = C(g)(\mathrm{e}^{a'}(k')^{-1}\sigma)$. By Lemma 2.9, $\mathrm{e}^a = g\,\mathrm{e}^{a'}\,g^{-1}$, $k^{-1}\sigma = g(k')^{-1}\sigma g^{-1}$, thus from Theorem 2.7, we get

$$y = pg \in X(\mathrm{e}^a) \cap X(k^{-1}\sigma). \tag{2.62}$$

Then

$$X(\gamma\sigma) \subset X(\mathrm{e}^a) \cap X(k^{-1}\sigma). \tag{2.63}$$

If $y = pg \in X(\mathrm{e}^a) \cap X(k^{-1}\sigma)$. By Theorem 2.7, there exist $a' \in \mathfrak{p}$, $k_1, k_2 \in K$ such that

$$\mathrm{e}^a = C(g)(\mathrm{e}^{a'}\,k_1^{-1})\,,\ \mathrm{Ad}(k_1)a' = a'\,,\ k^{-1} = C^\sigma(g)(k_2^{-1}). \tag{2.64}$$

By (2.38), (2.49), we have $k_2^{-1}\sigma \in C(g^{-1})\widetilde{Z}(\mathrm{e}^a) = \widetilde{Z}(a') \cap \widetilde{Z}(k_1)$. Put $k' = k_2k_1 \in K$, then $\mathrm{e}^a\,k^{-1}\sigma = g\,\mathrm{e}^{a'}(k')^{-1}\sigma g^{-1}$ with $\mathrm{Ad}(k')a' = \sigma a'$. Thus $y = pg \in X(\mathrm{e}^a\,k^{-1}\sigma) = X(\gamma\sigma)$. This proves (2.57).

The first identification in (2.58) is proved in [9, Theorem 3.2.6]. We only prove the second one. Clearly, $\mathfrak{p}_\sigma(k^{-1}) \subset X(k^{-1}\sigma)$ under the coordinate $(\mathfrak{p}, \exp_{x_0})$. If $b \in \mathfrak{p}$ is such that $\exp_{x_0}(b) \in X(k^{-1}\sigma)$, then there exists $k' \in K$ such that

$$k^{-1}\exp(\sigma(b)) = \exp(b)k'. \tag{2.65}$$

We can rewrite (2.65) as

$$\exp\big(\mathrm{Ad}(k^{-1})\sigma(b)\big)k^{-1} = \exp(b)k'. \tag{2.66}$$

Then we get

$$\mathrm{Ad}(k^{-1}\sigma)b = b\,,\ k' = k^{-1}. \tag{2.67}$$

This means exactly $b \in \mathfrak{p}_\sigma(k^{-1})$. Then the proof to (2.58) is completed. □

Remark 2.12. — Note that θ acts on $Z^0_\sigma(\gamma)$ as an automorphism so that $Z^0_\sigma(\gamma)$ is a real reductive Lie group, in the sense of [29, §7.2], equipped with the Cartan involution $\theta|_{Z^0_\sigma(\gamma)}$ and the invariant bilinear form $B|_{\mathfrak{z}_\sigma(\gamma)}$.

By (2.60), as in (2.22), the tangent bundle of $X(\gamma\sigma)$ is given as follows

$$TX(\gamma\sigma) = Z_\sigma(\gamma) \times_{K_\sigma(\gamma)} \mathfrak{p}_\sigma(\gamma). \tag{2.68}$$

Let $\mathfrak{z}^\perp_\sigma(\gamma)$ be the orthogonal subspace of $\mathfrak{z}_\sigma(\gamma)$ in $\mathfrak{g}$ with respect to B. Put

$$\mathfrak{p}^\perp_\sigma(\gamma) = \mathfrak{z}^\perp_\sigma(\gamma) \cap \mathfrak{p},\ \mathfrak{k}^\perp_\sigma(\gamma) = \mathfrak{z}^\perp_\sigma(\gamma) \cap \mathfrak{k}. \tag{2.69}$$

By (2.55), we get

$$\mathfrak{z}^\perp_\sigma(\gamma) = \mathfrak{p}^\perp_\sigma(\gamma) \oplus \mathfrak{k}^\perp_\sigma(\gamma). \tag{2.70}$$

If $N_{X(\gamma\sigma)/X}$ is the normal vector bundle of $X(\gamma\sigma)$ in X, by (2.68), then

$$N_{X(\gamma\sigma)/X} = Z_\sigma(\gamma) \times_{K_\sigma(\gamma)} \mathfrak{p}^\perp_\sigma(\gamma). \tag{2.71}$$

Let $\mathcal{N}_{X(\gamma\sigma)/X}$ be the total space of $N_{X(\gamma\sigma)/X} \to X(\gamma\sigma)$.

Let $P_{\gamma\sigma} : X \to X(\gamma\sigma)$ be the orthogonal projection from X into $X(\gamma\sigma)$ [21, Proposition 1.6.3]. Due to the geometric structures established in Theorem 2.11, the same proof (using essentially the convexity of displacement functions) of [9, Theorems 3.4.1 and 3.4.3] gives the following estimates for the displacement function $d_{\gamma\sigma}$ along the normal vector space of $X(\gamma\sigma)$. It will guarantee the convergence of the twisted orbital integrals defined in Section 3.

Theorem 2.13. — *We have the diffeomorphism of $Z_\sigma(\gamma)$-manifolds,*

$$\rho_{\gamma\sigma} : (g, f) \in \mathcal{N}_{X(\gamma\sigma)/X} \longrightarrow p(g\exp(f)) \in X. \tag{2.72}$$

The action of $\gamma\sigma$ on X, through the above diffeomorphism, is represented by the map $(g, f) \mapsto (\exp(a)g, \mathrm{Ad}(k^{-1})\sigma(f))$, and the projection $P_{\gamma\sigma}$ is given by $P_{\gamma\sigma}(g, f) = (g, 0)$.

There exists $c_{\gamma\sigma} > 0$, such that if $f \in \mathfrak{p}^\perp_\sigma(\gamma)$, $|f| \geqslant 1$, then

$$d_{\gamma\sigma}(\rho_{\gamma\sigma}(1, f)) \geqslant |a| + c_{\gamma\sigma}|f|. \tag{2.73}$$

There exist $C'_{\gamma\sigma} > 0$, $C''_{\gamma\sigma} > 0$ such that, for $f \in \mathfrak{p}^\perp_\sigma(\gamma)$, if $|f| \geqslant 1$,

$$\left|\nabla d_{\gamma\sigma}(\rho_{\gamma\sigma}(1, f))\right| \geqslant C'_{\gamma\sigma}, \tag{2.74}$$

and if $|f| \leqslant 1$,

$$\left|\nabla d^2_{\gamma\sigma}(\rho_{\gamma\sigma}(1, f))/2\right| \geqslant C''_{\gamma\sigma}|f|. \tag{2.75}$$

The group $K_\sigma(\gamma)$ acts on the left on K and on $\mathfrak{p}_\sigma^\perp(\gamma)$. Let $\mathfrak{p}_\sigma^\perp(\gamma)_{K_\sigma(\gamma)} \times K$ be the vector bundle on $K_\sigma(\gamma)\backslash K$ given by the relation, for $f \in \mathfrak{p}_\sigma^\perp(\gamma)$, $k \in K$ and $h \in K_\sigma(\gamma)$,

$$(f,k) \sim (\mathrm{Ad}(h)f, hk). \tag{2.76}$$

Right multiplication by K lifts to $\mathfrak{p}_\sigma^\perp(\gamma)_{K_\sigma(\gamma)} \times K$. By Theorem 2.13, we get a diffeomorphism,

$$\varrho_{\gamma\sigma} : (g,f,k) \in Z_\sigma(\gamma) \times_{K_\sigma(\gamma)} (\mathfrak{p}_\sigma^\perp(\gamma) \times K) \to g\,\mathrm{e}^f\,k \in G. \tag{2.77}$$

As a consequence, we have

$$\mathfrak{p}_\sigma^\perp(\gamma)_{K_\sigma(\gamma)} \times K = Z_\sigma(\gamma)\backslash G. \tag{2.78}$$

Similarly, by taking the identity components of the twisted centralizers, we get

$$\mathfrak{p}_\sigma^\perp(\gamma)_{K^0_\sigma(\gamma)} \times K = Z^0_\sigma(\gamma)\backslash G. \tag{2.79}$$

Remark 2.14. — Let $Z^\sigma(\gamma\sigma)$, $K^\sigma(\gamma\sigma)$ denote the centralizers of $\gamma\sigma$ in G^σ, K^σ respectively. In Theorems 2.13 and (2.78), if we replace $Z_\sigma(\gamma)$, $K_\sigma(\gamma)$, G, and K by $Z^\sigma(\gamma\sigma)$, $K^\sigma(\gamma\sigma)$, G^σ, and K^σ respectively, we still have analogue results. The reason is that our proof to them relies on the identities obtained in Proposition 2.8 and Lemma 2.9, which holds for group G^σ, K^σ.

The following result is classical for linear algebraic groups, such as [27, 18.2 Proposition p. 117], [13, III. Theorem 9.2], [1, Chapter 1, p. 22], etc. Here in our setting, we reproduce a proof using the above geometric constructions.

Proposition 2.15. — *For $\gamma \in G$, the element $\gamma\sigma$ is semisimple if and only if the σ-conjugacy class of γ in G is a closed subset.*

Proof. — At first, we assume that $\gamma\sigma$ is semisimple, moreover, we may and we will assume that γ is given as in (2.50), and let $[\gamma]_\sigma \subset G$ denote the σ-conjugacy class of γ. Let $\{\gamma_i\}_{i\in\mathbb{N}} \subset [\gamma]_\sigma$ be a Cauchy sequence in G with the limit $h_0 \in G$. In particular, we have, as $i \to +\infty$,

$$d\big(p\gamma_i, ph_0\big) \to 0. \tag{2.80}$$

By (2.78), for $i \in \mathbb{N}$, there exists $g_i = \mathrm{e}^{f_i}\,k_i$, $f_i \in \mathfrak{p}_\sigma^\perp(\gamma)$, $k_i \in K$ such that

$$\gamma_i = g_i^{-1}\gamma\sigma(g_i). \tag{2.81}$$

Then we get, as $i \to +\infty$,

$$d\big(\gamma\sigma p\,\mathrm{e}^{f_i}, g_i p h_0\big) \to 0. \tag{2.82}$$

Using the triangle inequality for the distance d on X, by (2.82), we get, as $i \to +\infty$,

$$d_{\gamma\sigma}(\mathrm{e}^{f_i}) = d\big(\gamma\sigma p\,\mathrm{e}^{f_i}, p\,\mathrm{e}^{f_i}\,\big) \to d\big(p1, ph_0\big). \tag{2.83}$$

Then by the estimate (2.73), we get the set $\{f_i\}_{i\in\mathbb{N}}$ is a bounded set in $\mathfrak{p}_\sigma^\perp(\gamma)$. Then we can assume that there exist $f' \in \mathfrak{p}_\sigma^\perp(\gamma)$, $k' \in K$ such that, by extracting a sub-sequence, as $i \to +\infty$,

$$f_i \to f', \;\; k_i \to k'. \tag{2.84}$$

Put $g' = \mathrm{e}^{f'}\,k' \in \widetilde{G}$, then as $i \to +\infty$,

$$g_i \to g'. \tag{2.85}$$

By (2.81), we get

$$h_0 = (g')^{-1}\gamma\sigma(g') \in [\gamma]_\sigma, \tag{2.86}$$

so that $[\gamma]_\sigma$ is a closed subset of G.

Now we prove another direction, and assume that $[\gamma]_\sigma$ is a closed subset. For the semisimplicity of $\gamma\sigma$, it is enough to find $x \in X$ such that $d_{\gamma\sigma}(x) = m_{\gamma\sigma}$. Let $\{g_i\}_{i\in\mathbb{N}} \subset G$ be such that as $i \to +\infty$,

$$d_{\gamma\sigma}(pg_i) = d(pg_i, \gamma\sigma pg_i) \to m_{\gamma\sigma}. \tag{2.87}$$

Set $\gamma_i = g_i^{-1}\gamma\sigma(g_i) \in [\gamma]_\sigma$. Then (2.87) is equivalent to $d(p1, p\gamma_i) \to m_{\gamma\sigma}$. As a consequence, the set $\{\gamma_i\}$ lies in a bounded subset of G, hence there exists a subsequence $\{\gamma_{k_i}\}_{i\in\mathbb{N}}$ which converges to an element $h_0 \in G$ as $i \to +\infty$. The closedness of $[\gamma]_\sigma$ infers that $h_0 = g^{-1}\gamma\sigma(g)$ for some $g \in G$. Taking $x = pg \in X$, then $d_{\gamma\sigma}(x) = m_{\gamma\sigma}$ and $\gamma\sigma$ is semisimple by definition. This completes the proof of our proposition. □

2.6. The locally symmetric space Z

We fix $\sigma \in \Sigma$ and fix a discrete torsion-free cocompact subgroup $\Gamma \subset G$ such that $\sigma(\Gamma) = \Gamma$. The following lemma is given by [44, Lemmas 1 and 2].

Lemma 2.16. — *If $\gamma \in \Gamma$, then $\gamma\sigma \in \widetilde{G}$ is semisimple, and $\Gamma \cap Z_\sigma(\gamma)$ is a cocompact discrete subgroup of $Z_\sigma(\gamma)$.*

Definition 2.17. — *We denote by $[\Gamma]_\sigma$ the set of σ-twisted conjugacy classes in Γ. If $\gamma \in \Gamma$, let $\underline{[\gamma]}_\sigma$ be the σ-twisted conjugacy class of γ in Γ. If $\gamma\sigma$ is elliptic, we say that $\underline{[\gamma]}_\sigma$ is an elliptic class. Let $\underline{E}_\sigma$ be the set of elliptic classes in $[\Gamma]_\sigma$.*

The map $\gamma' \in \Gamma \mapsto (\gamma')^{-1}\gamma\sigma(\gamma') \in \underline{[\gamma]}_\sigma$ induces the identification

$$\underline{[\gamma]}_\sigma \simeq \Gamma \cap Z_\sigma(\gamma)\backslash\Gamma. \tag{2.88}$$

Lemma 2.18. — *The set $\underline{E}_\sigma$ is finite.*

Proof. — Let $U \subset G$ be a compact fundamental domain for the left action of Γ on G such that $G = \cup_{\gamma\in\Gamma}\gamma U$. Note that $p : G \to X$ is a proper map. Put

$$V = p^{-1}(p(U)) = U \cdot K. \tag{2.89}$$

Then V is a compact subset of G. We denote by V^{-1} the set of the inverses of elements in V, both V^{-1} and $V \cdot \sigma(V^{-1})$ are compact.

For any $\underline{[\gamma]}_\sigma \in \underline{E}_\sigma$, there exists $\gamma' \in \underline{[\gamma]}_\sigma$ such that $\gamma'\sigma$ has fixed points in $p(V) = p(U)$. Let $g_\gamma \in U$ be such that pg_γ is fixed by $\gamma'\sigma$. Then we get

$$\gamma' \in UK\sigma(U^{-1}) \cap \Gamma \subset V \cdot \sigma(V^{-1}) \cap \Gamma. \tag{2.90}$$

Since $V \cdot \sigma(V^{-1})$ is compact, $V \cdot \sigma(V^{-1}) \cap \Gamma$ is a finite set, and the lemma follows. □

Put $Z = \Gamma\backslash X = \Gamma\backslash G/K$, then Z is a compact locally symmetric manifold. The homogeneous vector bundle (F, h^F, ∇^F) on X defined in Subsection 2.1 descends to a vector bundle on Z, which we still denote by (F, h^F, ∇^F). In particular, the tangent bundle TX descends to the tangent bundle TZ, and N also descends to a Euclidean vector bundle, which we still denote it by N.

Since $\sigma(\Gamma) = \Gamma$. Then σ acts isometrically on Z. Let ${}^\sigma Z \subset Z$ be the fixed point set of σ in Z. If $g \in G$ (resp. $x \in X$), we denote by $[g]_Z$ (resp. $[x]_Z$) the corresponding point in Z.

Lemma 2.19. — *If $\gamma_1, \gamma_2 \in \Gamma$ are σ-twisted conjugate in Γ, then*

$$[X(\gamma_1\sigma)]_Z = [X(\gamma_2\sigma)]_Z \subset Z. \tag{2.91}$$

If $g \in G$, then $[g]_Z \in {}^\sigma Z$ if and only if there is $\gamma \in \Gamma$ such that $\gamma\sigma$ is elliptic and that $pg \in X(\gamma\sigma) \subset X$. If $\underline{[\gamma_1]}_\sigma$, $\underline{[\gamma_2]}_\sigma \in \underline{E}_\sigma$ are distinct classes, then

$$[X(\gamma_1\sigma)]_Z \cap [X(\gamma_2\sigma)]_Z = \emptyset. \tag{2.92}$$

Proof. — The first part of our lemma is clear. If $[g]_Z \in {}^\sigma Z$, then there are $\gamma_0 \in \Gamma$, $k_0 \in K$ such that

$$\sigma(g) = \gamma_0 g k_0. \tag{2.93}$$

Then $\gamma_0^{-1}\sigma(g) = gk_0$, so that $pg \in X$ is a fixed point of $\gamma_0^{-1}\sigma$, and $\gamma_0^{-1}\sigma$ is elliptic. If $x \in X$ and $\gamma\sigma(x) = x$ with some $\gamma \in \Gamma$, then $[x]_Z \in {}^\sigma Z$ by definition.

Suppose that $\underline{[\gamma_1]}_\sigma, \underline{[\gamma_2]}_\sigma \in \underline{E}_\sigma$. If $[X(\gamma_1\sigma)]_Z \cap [X(\gamma_2\sigma)]_Z \neq \emptyset$ in Z, since $\gamma_1\sigma, \gamma_2\sigma$ are elliptic, there exists $\gamma \in \Gamma$ and $x \in X$ such that

$$\gamma^{-1}\gamma_1\sigma(\gamma)\sigma(x) = \gamma_2\sigma(x) = x. \tag{2.94}$$

Then

$$\gamma_2^{-1}\gamma^{-1}\gamma_1\sigma(\gamma)\sigma(x) = \sigma(x).$$

Since Γ is torsion-free, then $\gamma_2 = \gamma^{-1}\gamma_1\sigma(\gamma)$, i.e., $\underline{[\gamma_1]}_\sigma = \underline{[\gamma_2]}_\sigma$. Then we get (2.92). This completes our proof. □

Using Lemma 2.19, we get that

$$^\sigma Z = \bigcup_{\underline{[\gamma]}_\sigma \in \underline{E}_\sigma} [X(\gamma\sigma)]_Z. \tag{2.95}$$

Moreover, the right-hand side in (2.95) is a finite disjoint union. By Lemma 2.16, $\Gamma \cap Z_\sigma(\gamma)$ is a cocompact torsion-free discrete subgroup of $Z_\sigma(\gamma)$, so that $\Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma)$ is a compact smooth manifold

Take $\underline{[\gamma]}_\sigma \in \underline{E}_\sigma$, let $\gamma \in \Gamma$ be one representative of $\underline{[\gamma]}_\sigma$. If $x \in X(\gamma\sigma)$, if $\gamma_0 \in \Gamma$ is such that $\gamma_0 x \in X(\gamma\sigma)$, then an argument like (2.94) gives that $\gamma_0 \in Z_\sigma(\gamma)$. Thus the projection $X \to Z$ induces an identification between $\Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma)$ and $[X(\gamma\sigma)]_Z \subset Z$. Then (2.95) can be rewritten as

$$^\sigma Z = \bigcup_{\underline{[\gamma]}_\sigma \in \underline{E}_\sigma} \Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma), \tag{2.96}$$

Let $C(Z,F)$ be the vector space of continuous sections of F on Z, which can be identified with the subspace of $C(X,F)$ of Γ-invariant sections over X, i.e.,

$$C(Z,F) = C(X,F)^\Gamma. \tag{2.97}$$

Then by (2.24), we get

$$C(Z,F) = C_K(G,E)^\Gamma. \tag{2.98}$$

Assume now that the vector bundle F is defined via a K^σ-representation (E,ρ^E). Since σ preserves Γ, the action of σ descends to $F \to Z$.

Proposition 2.20. — *Take $\underline{[\gamma]}_\sigma \in \underline{E}_\sigma$. Under the identification* (2.96)*, the action of σ on the bundle F restricted to $[X(\gamma\sigma)]_Z \subset {}^\sigma Z$ is given by the action of $\gamma\sigma$ on the vector bundle F over $\Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma)$.*

Proof. — Take $x_0 = pg_0 \in X(\gamma\sigma)$. There is $k \in K$ such that

$$\gamma = C^\sigma(g_0)(k^{-1}). \tag{2.99}$$

By Proposition 2.11 and (2.99), we have

$$X(\gamma\sigma) = g_0(X(k^{-1}\sigma)). \tag{2.100}$$

By (2.23), (2.100), we have the identification of vector bundles,

$$F|_{[X(\gamma\sigma)]_Z} \simeq \Gamma \cap Z_\sigma(\gamma)\backslash\, g_0\big(Z_\sigma(k^{-1}) \times_{K_\sigma(k^{-1})} E\big). \tag{2.101}$$

If $g \in Z_\sigma(k^{-1})$, by (2.99), we get

$$\sigma(g_0 g) = \gamma^{-1} g_0 g k^{-1}. \tag{2.102}$$

Put $x = p(g_0 g) \in X(\gamma\sigma)$ and $z = [g_0 g]_Z \in [X(\gamma\sigma)]_Z$. If $v \in F_z \simeq E$, then

$$\begin{aligned}\sigma(z,v) &= (\sigma(z), \sigma v)\\ &= [(\sigma(g_0 g), \rho^E(\sigma)v)]_Z\\ &= [(g_0 g, \rho^E(k^{-1}\sigma)v)]_Z \in F_{\sigma(z)}.\end{aligned} \tag{2.103}$$

Take the lift of $[(g_0 g, \rho^E(k^{-1}\sigma)v)]_Z$ around x, as $gk^{-1}\sigma = k^{-1}\sigma g$, we have

$$[(g_0 g, \rho^E(k^{-1}\sigma^E)v)]_Z = g_0 k^{-1}\sigma g_0^{-1}(x,v) = \gamma\sigma(x,v). \tag{2.104}$$

This completes the proof of our proposition. □

3. The twisted orbital integrals

In this section, we give a geometric interpretation for the twisted orbital integrals associated with a semisimple element in $\widetilde{G}$. The constructions given here generalize the results of [9, Chapter 4]. We fix one element $\sigma \in \Sigma$.

3.1. An algebra of invariant kernels on X

Recall that (E, ρ^E) is a unitary representation of K^σ, and that $F = G \times_K E$ is the associated Hermitian vector bundle on X. We introduce a vector space $\mathcal{Q}^\sigma$ of continuous invariant kernels as follows.

Definition 3.1. — *Let $\mathcal{Q}^\sigma$ be the vector space of maps $q \in C(G, \mathrm{End}(E))$ which satisfy that*

- *There exist $C, C' > 0$, such that*

$$|q(g)| \leqslant C \exp\left(-C' d^2(p1, pg)\right),\ \forall\, g \in G. \tag{3.1}$$

- *For $k, k' \in K$, we have*

$$q(kgk') = \rho^E(k) q(g) \rho^E(k'). \tag{3.2}$$

- *Set* $\sigma^E = \rho^E(\sigma) \in \mathrm{Aut}(E)$,

$$q(\sigma(g)) = \sigma^E q(g)(\sigma^E)^{-1} \in \mathrm{End}(E). \tag{3.3}$$

Let $C^b(G,E)$ be the set of bounded continuous functions on G valued in E. For $q \in \mathcal{Q}^\sigma$ and $g, g' \in G$, put

$$q(g,g') = q(g^{-1}g') \in \mathrm{End}(E). \tag{3.4}$$

By (3.2), $q(g,g')$ defines an integral operator Q acting on $C^b(G,E)$, which is K-equivariant. Then it descends to an operator acting on $C^b(X,F)$. Let $q(x,x') \in \mathrm{Hom}(F_{x'}, F_x)$ be the corresponding continuous kernel on $X \times X$, which is just the descent of $q(g,g')$ to $X \times X$. Moreover, the condition (3.3) is equivalent to say that, for $x, x' \in X$,

$$q^X(\sigma(x), \sigma(x')) = \sigma q^X(x,x')\sigma^{-1} \in \mathrm{Hom}(F_{\sigma(x')}, F_{\sigma(x)}). \tag{3.5}$$

By (3.4), and (3.5), Q commutes with G^σ-action.

Remark 3.2. — The vector space $\mathcal{Q}^\sigma$ with the convolution of kernel functions becomes an associative algebra, it is a subalgebra of the one defined in [9, Definition 4.1.1].

We can extend $q \in \mathcal{Q}^\sigma$ to a continuous map $\widetilde{q} \in C(G^\sigma, \mathrm{End}(E))$ by setting

$$\widetilde{q}(g\mu) = q(g)\rho^E(\mu) \in \mathrm{End}(E)\ ,\ g \in G, \mu \in \Sigma^\sigma. \tag{3.6}$$

Then it lifts to a continuous kernel defined on $G^\sigma \times G^\sigma$ such that

$$\widetilde{q}(g\mu, h\mu') = \widetilde{q}((g\mu)^{-1}h\mu') \in \mathrm{End}(E). \tag{3.7}$$

The operator Q can be also expressed as the integral operator on $C^b_{K^\sigma}(G^\sigma, E)$ associated with kernel $\widetilde{q}$.

Since we are going to define the twisted orbital integral in next subsection, we need to introduce the volume measures which are involved here. Let $\mathrm{d}x$ be the volume element on X induced by the Riemannian metric. Let $\mathrm{d}k$ be the normalized Haar measure of K. Put

$$\mathrm{d}g = \mathrm{d}x\,\mathrm{d}k. \tag{3.8}$$

Then $\mathrm{d}g$ is a left-invariant Haar measure on G. Since G is unimodular, $\mathrm{d}g$ is also right-invariant.

Let $\mathrm{d}y$ be the volume element on $X(\gamma\sigma)$ induced by Riemannian metric, let $\mathrm{d}f$ be the volume element on the Euclidean space $\mathfrak{p}^\perp_\sigma(\gamma)$. Then $\mathrm{d}y\,\mathrm{d}f$ is a volume element on $Z_\sigma(\gamma) \times_{K_\sigma(\gamma)} \mathfrak{p}^\perp_\sigma(\gamma)$ which is $Z_\sigma(\gamma)$-invariant. By

Theorem 2.13, there is a smooth positive $K_\sigma(\gamma)$-invariant function $r(f)$ on $\mathfrak{p}_\sigma^\perp(\gamma)$ such that we have the identity of volume elements on X,

$$\mathrm{d}x = r(f)\,\mathrm{d}y\,\mathrm{d}f, \tag{3.9}$$

with $r(0) = 1$. Moreover, there exist $C > 0,\ C' > 0$ such that for $f \in \mathfrak{p}_\sigma^\perp(\gamma)$ (cf. [22, Chapter IV, Theorem 4.1]),

$$r(f) \leqslant C \exp\left(C'|f|\right). \tag{3.10}$$

Let $\mathrm{d}k'$ be the Haar measure on $K_\sigma(\gamma)$ that gives volume 1 to $K_\sigma(\gamma)$, and let $\mathrm{d}u$ be the K-invariant volume form on $K_\sigma(\gamma)\backslash K$, so that

$$\mathrm{d}k = \mathrm{d}k'\mathrm{d}u. \tag{3.11}$$

Then $\mathrm{d}y\,\mathrm{d}k'$ defines an invariant Haar measure on the reductive Lie group $Z_\sigma(\gamma)$ such that

$$\mathrm{d}g = \mathrm{d}y\,\mathrm{d}k' \cdot r(f)\,\mathrm{d}f\,\mathrm{d}u. \tag{3.12}$$

By (2.78), (3.12), $\mathrm{d}v = r(f)\,\mathrm{d}f\,\mathrm{d}u$ is a G-invariant measure on $Z_\sigma(\gamma)\backslash G$.

3.2. Twisted orbital integrals

Let $\gamma \in G$ be as follows

$$\gamma = \mathrm{e}^a\, k^{-1} \in G,\ a \in \mathfrak{p},\ k \in K,\ \mathrm{Ad}(k)a = \sigma a. \tag{3.13}$$

Then $\gamma\sigma$ is semisimple.

If $q \in \mathcal{Q}^\sigma$, then for $x \in X$, we have $\gamma\sigma q\big(x, \gamma\sigma(x)\big) \in \mathrm{End}(F_{\gamma\sigma(x)})$. Therefore, $\mathrm{Tr}^F\left[\gamma\sigma q\big(x, \gamma\sigma(x)\big)\right]$ is well-defined function on X. Let $h(y)$ be a compactly supported bounded measurable function on $X(\gamma\sigma)$.

Proposition 3.3. — *The function* $\mathrm{Tr}^F\left[\gamma\sigma q\big(x, \gamma\sigma(x)\big)\right] h(p_{\gamma\sigma}x)$ *is integrable on* X*. Moreover,*

$$\int_X \mathrm{Tr}^F\left[\gamma\sigma q\big(x, \gamma\sigma(x)\big)\right] h(p_{\gamma\sigma}x)\,\mathrm{d}x = \int_{\mathfrak{p}_\sigma^\perp(\gamma)} \mathrm{Tr}^E\left[\sigma^E q(\mathrm{e}^{-f}\,\gamma\,\mathrm{e}^{\sigma f})\right] r(f)\,\mathrm{d}f \int_{X(\gamma\sigma)} h(y)\,\mathrm{d}y. \tag{3.14}$$

Proof. — By (2.73) and (3.1), the function $\mathrm{Tr}^E[\sigma^E q(\mathrm{e}^{-f}\,\gamma\,\mathrm{e}^{\sigma f})]$ is bounded by $C'\exp(-C|f|^2)$ with some constants $C, C' > 0$ for $f \in \mathfrak{p}_\sigma^\perp(\gamma)$. By (3.10), the integrals in the right-hand side of (3.14) are well-defined. By the identification $\rho_{\gamma\sigma}$ defined in (2.72) and using the Fubini's theorem, we get exactly (3.14). □

By (2.78), and using the fact that the Haar measures of K, $K_\sigma(\gamma)$ have volume 1, we have

$$\begin{aligned}(3.15)\quad \int_{\mathfrak{p}_\sigma^\perp(\gamma)} \operatorname{Tr}^E\left[\sigma^E q(\mathrm{e}^{-f}\,\gamma\,\mathrm{e}^{\sigma f})\right] r(f)\,\mathrm{d}f \\ = \int_{Z_\sigma(\gamma)\backslash G} \operatorname{Tr}^E\left[\sigma^E q\big(v^{-1}\gamma\sigma(v)\big)\right]\mathrm{d}v.\end{aligned}$$

Recall that $Z^\sigma(\gamma\sigma)$ denotes the centralizer of $\gamma\sigma$ in G^σ. As said in Remark 2.14, an analogue of (2.78) for the pair $(G^\sigma, K^\sigma, Z^\sigma(\gamma\sigma))$ still holds. Then the above integrals can be rewritten as integrals on the quotient $Z^\sigma(\gamma\sigma)\backslash G^\sigma$ with the kernel $\widetilde{q}$ defined in (3.6). More precisely, put

$$(3.16)\qquad \mathrm{d}\widetilde{k} = \mathrm{d}k\,\mathrm{d}\mu.$$

Then $\mathrm{d}\widetilde{k}$ is the normalized Haar measure on K^σ. Let $\mathrm{d}\widetilde{k}^\sigma$ be the normalized Haar measure on $K^\sigma(\gamma\sigma)$, and let $\mathrm{d}\widetilde{\mu}^\sigma$ be the K^σ-invariant measure on $K^\sigma(\gamma\sigma)\backslash K^\sigma$ such that

$$(3.17)\qquad \mathrm{d}\widetilde{k} = \mathrm{d}\widetilde{k}^\sigma \mathrm{d}\widetilde{\mu}^\sigma.$$

Then by (2.78), we get that

$$(3.18)\qquad \mathrm{d}\widetilde{v}^\sigma = r(f)\,\mathrm{d}f\,\mathrm{d}\widetilde{\mu}^\sigma$$

is a measure on $Z^\sigma(\gamma\sigma)\backslash G^\sigma$. Then

$$(3.19)\quad \int_{\mathfrak{p}_\sigma^\perp(\gamma)} \operatorname{Tr}^E[\sigma^E q(\mathrm{e}^{-f}\gamma\mathrm{e}^{\sigma f})] r(f)\,\mathrm{d}f = \int_{Z^\sigma(\gamma\sigma)\backslash G^\sigma} \operatorname{Tr}^E[\widetilde{q}(\widetilde{v}^{-1}\gamma\sigma\widetilde{v}]\,\mathrm{d}\widetilde{v}^\sigma.$$

Let $[\gamma\sigma]$ denote the conjugation class of $\gamma\sigma$ in G^σ.

Definition 3.4. — *For $q \in \mathcal{Q}^\sigma$, set*

$$(3.20)\quad \begin{aligned}\operatorname{Tr}^{[\gamma\sigma]}[Q] &= \int_{Z_\sigma(\gamma)\backslash G} \operatorname{Tr}^E\left[\sigma^E q(v^{-1}\gamma\sigma(v))\right]\mathrm{d}v \\ &= \int_{\mathfrak{p}_\sigma^\perp(\gamma)} \operatorname{Tr}^E\left[\sigma^E q(\mathrm{e}^{-f}\,\gamma\,\mathrm{e}^{\sigma f})\right] r(f)\,\mathrm{d}f.\end{aligned}$$

Integrals like (3.15), (3.19), (3.20) *are called twisted orbital integrals. By* (3.19), *we see that* $\operatorname{Tr}^{[\gamma\sigma]}[Q]$ *only depends on the conjugacy class of $\gamma\sigma$ in G^σ. In particular, if $\gamma' \in G$ is σ-twisted conjugate to γ, then* $\operatorname{Tr}^{[\gamma'\sigma]}[Q] = \operatorname{Tr}^{[\gamma\sigma]}[Q]$.

If taking $\sigma = \mathrm{Id}_G$ in (3.20), *we get the ordinary (un-twisted) orbital integral* $\operatorname{Tr}^{[\gamma]}[Q]$ *(cf.* [9, *Definition 4.2.2*]*) associated with a semisimple element $\gamma \in G$.*

The following proposition extends [9, Theorem 4.2.3].

PROPOSITION 3.5. — *For $Q, Q' \in \mathcal{Q}^\sigma$, we have*

$$\mathrm{Tr}^{[\gamma\sigma]}\left[[Q,Q']\right] = 0. \tag{3.21}$$

Equivalently, $\mathrm{Tr}^{[\gamma\sigma]}[\cdot]$ *is a trace on the algebra $\mathcal{Q}^\sigma$.*

Proof. — Let R be an integral operator defined by a kernel function in $\mathcal{Q}^\sigma$, and let R' be an integral operator associated with a bounded continuous invariant kernel function in $C^b(G^\sigma, \mathrm{End}(E))$. They act on continuous sections of F over X with compact support. The operators RR', $R'R$ also have integral kernels which are bounded on G^σ. We have

$$\mathrm{Tr}^{[1]}\left[[R,R']\right] = 0. \tag{3.22}$$

Let $\delta_{\gamma\sigma}$ be the current on G^σ so that

$$\int_{G^\sigma} f\delta_{\gamma\sigma} = \int_{Z^\sigma(\gamma\sigma)\backslash G^\sigma} f((\widetilde{v})^{-1}\gamma\sigma\widetilde{v})\,\mathrm{d}\widetilde{v}^\sigma. \tag{3.23}$$

Since $\mathrm{d}\widetilde{v}^\sigma$ is invariant under the right action of G^σ on $Z^\sigma(\gamma\sigma)\backslash G^\sigma$, $\delta_{\gamma\sigma}$ is invariant by conjugation. For $q \in \mathcal{Q}^\sigma$, $\widetilde{q}$ is defined by (3.6). By (3.19), (3.20),

$$\mathrm{Tr}^{[\gamma\sigma]}[Q] = \int_{G^\sigma} \mathrm{Tr}^E[\widetilde{q}]\delta_{\gamma\sigma} = \mathrm{Tr}^E[\widetilde{q} * \delta_{(\gamma\sigma)^{-1}}(1)], \tag{3.24}$$

where $*$ denotes the convolution on G^σ.

The current $\delta_{(\gamma\sigma)^{-1}}$ defines a linear operator $R_{(\gamma\sigma)^{-1}}$ acting on $C^b(X,F)$. If $Q \in \mathcal{Q}^\sigma$, we have

$$QR_{(\gamma\sigma)^{-1}} = R_{(\gamma\sigma)^{-1}}Q. \tag{3.25}$$

The operator $QR_{(\gamma\sigma)^{-1}}$ is an integral operator with a bounded continuous invariant kernel.

Then we can rewrite (3.24) as

$$\mathrm{Tr}^{[\gamma\sigma]}[Q] = \mathrm{Tr}^{[1]}[QR_{(\gamma\sigma)^{-1}}]. \tag{3.26}$$

By (3.22), (3.25) and (3.26), we get

$$\mathrm{Tr}^{[\gamma\sigma]}\left[[Q,Q']\right] = \mathrm{Tr}^{[1]}\left[[Q,Q']R_{(\gamma\sigma)^{-1}}\right] = \mathrm{Tr}^{[1]}\left[[Q,Q'R_{(\gamma\sigma)^{-1}}]\right] = 0. \tag{3.27}$$

This completes the proof of our proposition. □

3.3. Infinite dimensional orbital integrals

Let $\mathrm{d}Y^{\mathfrak{p}}$, $\mathrm{d}Y^{\mathfrak{k}}$ denote the volume elements on the Euclidean spaces $\mathfrak{p}$, $\mathfrak{k}$. Then these volume elements are K^σ-invariant. Moreover, $\mathrm{d}Y = \mathrm{d}Y^{\mathfrak{p}}\mathrm{d}Y^{\mathfrak{k}}$ is a G^σ-invariant volume element on $\mathfrak{g}$. Let $\mathrm{d}Y^{TX}, \mathrm{d}Y^N, \mathrm{d}Y$ be the corresponding volume elements on the fibres of $TX, N, TX \oplus N$ over X.

Let $C^{\infty,b}(\mathfrak{g},\mathbb{R})$ be the vector space of real valued smooth bounded functions on $\mathfrak{g}$. We replace the finite-dimensional vector space E by the infinite dimensional space

$$\mathcal{E} = \Lambda^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes C^{\infty,b}(\mathfrak{g},\mathbb{R}) \otimes E$$

equipped with the natural action of K^σ. Then the vector bundle F on X is replaced by

$$\mathcal{F} = \Lambda^\bullet(T^*X \oplus N^*) \otimes C^{\infty,b}(TX \oplus N,\mathbb{R}) \otimes F. \tag{3.28}$$

Let $C^b(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F))$ be the space of continuous bounded sections of $\widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F)$ over $\widehat{\mathcal{X}}$.

The group K^σ acts on $C^b(G^\sigma \times \mathfrak{g}, \Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E)$, so that if $s \in C^b(G^\sigma \times \mathfrak{g}, \Lambda^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E)$ then for $\widetilde{k} \in K^\sigma$

$$(\widetilde{k} \cdot s)(\widetilde{g}, Y) = \rho^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}(\widetilde{k}) s(\widetilde{g}\widetilde{k}, \mathrm{Ad}(\widetilde{k}^{-1})Y). \tag{3.29}$$

Let $C^b_{K^\sigma}(G^\sigma \times \mathfrak{g}, \Lambda^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E)$ be the vector space of K^σ-invariant continuous bounded function on $G^\sigma \times \mathfrak{g}$ with values in $\Lambda^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E$. Then we have

$$\begin{aligned} C^b\big(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F)\big) &= C^b_{K^\sigma}\big(G^\sigma \times \mathfrak{g}, \Lambda^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E\big) \\ &= C^b_K\big(G \times \mathfrak{g}, \Lambda^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E\big). \end{aligned} \tag{3.30}$$

Definition 3.6. — *Let $\mathfrak{Q}^\sigma$ be the vector space of continuous kernels $q(g,Y,Y')$ defined on $G \times \mathfrak{g} \times \mathfrak{g}$ with values in* $\mathrm{End}(\Lambda^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E)$ *such that*

- *If $g \in G, k, k' \in K, Y, Y' \in \mathfrak{g}$, then*

$$\begin{aligned} &q\big(kgk', Y, Y'\big) \\ &\quad = \rho^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}(k) q\big(g, \mathrm{Ad}(k^{-1})Y, \mathrm{Ad}(k')Y'\big)\rho^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}(k'). \end{aligned} \tag{3.31}$$

- *Put $\sigma^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} = \rho^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}(\sigma) \in \mathrm{Aut}(\Lambda^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E)$, then*

$$q(\sigma(g), \sigma Y, \sigma Y') = \sigma^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q(g, Y, Y') (\sigma^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E})^{-1}. \tag{3.32}$$

- *There exist $C, C' > 0$ such that*

$$|q(g,Y,Y')| \leqslant C \exp\Big(-C'\big(d^2(p1,pg) + |Y|^2 + |Y'|^2\big)\Big). \tag{3.33}$$

We will denote by $\mathfrak{Q}^{\sigma,\infty}$ the subspace of $\mathfrak{Q}^{\sigma}$ consisting of smooth kernels.

If $q \in \mathfrak{Q}^{\sigma}$, put $q((g,Y),(g',Y')) = q(g^{-1}g', Y, Y')$. If $s \in C^b_K(G \times \mathfrak{g}, \Lambda^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E)$, put

$$(Qs)(g,Y) = \int_{G\times\mathfrak{g}} q\big((g,Y),(g',Y')\big)s(g',Y')\mathrm{d}g'\mathrm{d}Y'. \tag{3.34}$$

By (3.31), (3.33), Q is an operator acting on $C^b_K(G \times \mathfrak{g}, \Lambda^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E)$. Equivalently, the operator Q acts on $C^b(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F))$ with a continuous kernel $q((x,Y),(x',Y'))$.

The action of σ on $C^b_K(G \times \mathfrak{g}, \Lambda^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E)$ is represented by

$$(\sigma s)(g,Y) = \sigma^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} s(\sigma^{-1}(g), \sigma^{-1}Y). \tag{3.35}$$

Then (3.32) is equivalent to $Q\sigma = \sigma Q$. Therefore, Q commutes with G^σ.

By [9, Proposition 4.3.2] and using the fact that σ preserves $\mathrm{d}x\mathrm{d}Y$, $\mathfrak{Q}^\sigma$ is an algebra with respect to the convolution of kernels. Let $[\cdot,\cdot]$ denote the supercommutator with respect to the $\mathbb{Z}_2$-graded structure of $\mathrm{End}(\Lambda^\bullet(\mathfrak{p}^* \oplus \mathfrak{k}^*)\otimes E)$, and let $\mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}[\cdot]$ be the supertrace on $\mathrm{End}(\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E)$.

If $g \in G$, let $q(g)$ be the operator on $\mathcal{E}$ defined by the kernel $q(g,Y,Y')$. Let $\sigma^{\mathcal{E}} \in \mathrm{End}(\mathcal{E})$ denote the action of σ on $\mathcal{E}$. Then for $g \in G$, the integral operator $\sigma^{\mathcal{E}} q(g^{-1}\gamma\sigma(g))$ acting on $\mathcal{E}$ is given by the continuous kernel $\sigma^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q(g^{-1}\gamma\sigma(g), \sigma^{-1}Y, Y')$ on $\mathfrak{g}\times\mathfrak{g}$. By (3.33), the function $\mathfrak{g} \ni Y \mapsto \mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}[\sigma^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q(g^{-1}\gamma\sigma(g), \sigma^{-1}Y, Y)]$ is integrable on $\mathfrak{g}$.

If $\sigma^{\mathcal{E}} q(g^{-1}\gamma\sigma(g))$ is trace class, by [19, Proposition 3.1.1], we get

$$\begin{aligned}\mathrm{Tr_s}^{\mathcal{E}}\left[\sigma^{\mathcal{E}} q(g^{-1}\gamma\sigma(g))\right] \\ = \int_{\mathfrak{g}} \mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}\left[\sigma^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q\left(g^{-1}\gamma\sigma(g), \sigma^{-1}Y, Y\right)\right]\mathrm{d}Y.\end{aligned} \tag{3.36}$$

Remark 3.7. — A sufficient condition for our operator to be trace class is that the kernel together with its derivatives in Y, Y' of arbitrary orders lie in the Schwartz space of $\mathfrak{g}\times\mathfrak{g}$.

By (3.33), there exists $C_{\gamma\sigma} > 0$ such that

$$\begin{aligned}\left|\int_{\mathfrak{g}} \mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}\left[\sigma^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q(g^{-1}\gamma\sigma(g), \sigma^{-1}Y, Y)\right]\mathrm{d}Y\right| \\ \leqslant C_{\gamma\sigma}\exp\left(-C' d^2(pg, \gamma\sigma pg)\right).\end{aligned} \tag{3.37}$$

By (2.73), the arguments in the proof of Proposition 3.3 show that the left-hand side of (3.37) is integrable on $\mathfrak{p}^\perp_\sigma(\gamma)$.

We extend the notion of the twisted orbital integrals to the infinite dimensional case, which is a twisted version of [9, Definition 4.3.3].

Definition 3.8. — *If Q is given by $q \in \mathfrak{Q}^\sigma$, we define $\mathrm{Tr_s}^{[\gamma\sigma]}[Q]$ as follows,*

$$\mathrm{Tr_s}^{[\gamma\sigma]}[Q] \tag{3.38}$$
$$= \int_{(Z_\sigma(\gamma)\backslash G)\times\mathfrak{g}} \mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}\left[\sigma^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q(v^{-1}\gamma\sigma(v), Y, \sigma Y)\right] \mathrm{d}v\, \mathrm{d}Y$$
$$= \int_{\mathfrak{p}_\sigma^\perp(\gamma)\times\mathfrak{g}} \mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}\left[\sigma^{\Lambda^\bullet(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q(\mathrm{e}^{-f}\,\gamma\,\mathrm{e}^{\sigma f}, Y, \sigma Y)\right] r(f)\, \mathrm{d}f\, \mathrm{d}Y.$$

Expressions such as (3.38) *are called twisted orbital supertraces.*

If $\sigma^{\mathcal{E}} q(g^{-1}\gamma\sigma(g))$ is trace class for $g \in G$, then we can rewrite (3.38) as

$$\mathrm{Tr_s}^{[\gamma\sigma]}[Q] = \int_{\mathfrak{p}_\sigma^\perp(\gamma)} \mathrm{Tr_s}^{\mathcal{E}}[\sigma^{\mathcal{E}} q(\mathrm{e}^{-f}\,\gamma\,\mathrm{e}^{\sigma f})] r(f)\, \mathrm{d}f. \tag{3.39}$$

Proposition 3.9. — *If $Q, Q' \in \mathfrak{Q}^\sigma$, then*

$$\mathrm{Tr_s}^{[\gamma\sigma]}\left[[Q, Q']\right] = 0. \tag{3.40}$$

Proof. — By the above constructions, the proof of our proposition is just a modification of the proof of Proposition 3.5. □

3.4. Twisted trace formula for locally symmetric spaces

Let Γ be a cocompact torsion-free discrete subgroup of G such that $\sigma(\Gamma) = \Gamma$. We still assume that F is associated with a finite-dimensional unitary representation (E, ρ^E) of K^σ. Put $Z = \Gamma\backslash X = \Gamma\backslash G/K$. We use the notation in Subsection 2.6. Recall that Σ^σ acts isometrically on Z and its action lifts to an action on the bundles TZ, F.

Let $\mathrm{d}z$ be the volume element of Z induced by the Riemannian metric. We still denote by $\mathrm{d}g$ the volume element on $\Gamma\backslash G$ induced by $\mathrm{d}g$.

Let Q be an operator with kernel $q \in \mathcal{Q}^\sigma$. Then Q descends to an operator Q^Z acting on $C(Z, F)$. Let $q^Z(z, z')$, $z, z' \in Z$ be the continuous kernel of Q^Z over Z. We also denote by z, z' their arbitrary lifts in X. Then

$$q^Z(z, z') = \sum_{\gamma\in\Gamma} \gamma q^X(\gamma^{-1}z, z') = \sum_{\gamma\in\Gamma} q^X(z, \gamma z')\gamma. \tag{3.41}$$

The convergence of the above sums are guaranteed by the cocompactness of Γ and the condition (3.1) for q.

Note that σ acts on $C^\infty(Z,F)$, we will denote it by σ^Z. Then σQ descends to $\sigma^Z Q^Z$. By (3.5), (3.41), the kernel of $\sigma^Z Q^Z$ is given by

$$(\sigma^Z Q^Z)(z,z') = \sum_{\gamma\in\Gamma} q^X(z,\gamma\sigma(z'))\gamma\sigma. \tag{3.42}$$

By (2.98), $(\sigma^Z Q^Z)(z,z')$ lifts to $G\times G$, so that

$$(\sigma^Z Q^Z)(g,g') = \sum_{\gamma\in\Gamma} q(g^{-1}\gamma\sigma(g'))\sigma^E \in \mathrm{End}(E). \tag{3.43}$$

Assume that Q^Z is trace class, so is $\sigma^Z Q^Z$, then

$$\begin{aligned} \mathrm{Tr}[\sigma^Z Q^Z] &= \int_Z \mathrm{Tr}^F\left[(\sigma^Z Q^Z)(z,z)\right]\mathrm{d}z \\ &= \int_{\Gamma\backslash G} \mathrm{Tr}^E\left[(\sigma^Z Q^Z)(g,g)\right]\mathrm{d}g. \end{aligned} \tag{3.44}$$

By Lemma 2.16, if $\gamma\in\Gamma$, $\Gamma\cap Z_\sigma(\gamma)\backslash Z_\sigma(\gamma)$ is a compact smooth manifold. Recall that $[\Gamma]_\sigma$ is the set of σ-twisted conjugacy classes in Γ. The twisted trace formula is given as follows,

$$\mathrm{Tr}[\sigma^Z Q^Z] = \sum_{\underline{[\gamma]}_\sigma\in[\Gamma]_\sigma} \mathrm{Vol}\left(\Gamma\cap Z_\sigma(\gamma)\backslash X(\gamma\sigma)\right)\mathrm{Tr}^{[\gamma\sigma]}[Q]. \tag{3.45}$$

4. A formula for semisimple twisted orbital integrals

The purpose of this section is to present the main results in this paper. We get an explicit geometric formula for the twisted orbital integrals associated with heat kernels of the Casimir operator, which is an extension of Bismut's formula for semisimple orbital integrals [9, Theorem 6.1.1]. The proof of this formula is deferred to Section 6. In the last subsection, we will apply our formula and explain its consequences in typical examples from cyclic base change theory.

4.1. The J-function $J_{\gamma\sigma}(Y_0^{\mathfrak{k}})$ on $\mathfrak{k}_\sigma(\gamma)$

The function $\widehat{A}(x)$ is given by

$$\widehat{A}(x) = \frac{x/2}{\sinh(x/2)}. \tag{4.1}$$

Let H be a finite-dimensional Hermitian vector space. If $B \in \mathrm{End}(H)$ is self-adjoint, then $\frac{B/2}{\sinh(B/2)}$ is a self-adjoint positive endomorphism. Put

$$\widehat{A}(B) = \det{}^{1/2}\left[\frac{B/2}{\sinh(B/2)}\right]. \tag{4.2}$$

If $\gamma \in G$, $\sigma \in \Sigma$ are such that $\gamma\sigma$ is semisimple, as in Sections 2 and 3, we may and we will assume that $\gamma = \mathrm{e}^a k^{-1}$ with $a \in \mathfrak{p}, k \in K$, and $\mathrm{Ad}(k)a = \sigma a$.

Set $\mathfrak{z}_0 = \mathfrak{z}(a) = \mathfrak{p}_0 \oplus \mathfrak{k}_0$, then $\mathfrak{z}_\sigma(\gamma)$ is a Lie subalgebra of $\mathfrak{z}_0$. Let $\mathfrak{z}_0^\perp$, $\mathfrak{p}_0^\perp$, $\mathfrak{k}_0^\perp$ be the orthogonal vector spaces to $\mathfrak{z}_0$, $\mathfrak{p}_0$, $\mathfrak{k}_0$ in $\mathfrak{g}, \mathfrak{p}, \mathfrak{k}$, and let $\mathfrak{z}_{\sigma,0}^\perp(\gamma)$, $\mathfrak{p}_{\sigma,0}^\perp(\gamma)$, $\mathfrak{k}_{\sigma,0}^\perp(\gamma)$ be the orthogonal spaces to $\mathfrak{z}_\sigma(\gamma)$, $\mathfrak{p}_\sigma(\gamma)$, $\mathfrak{k}_\sigma(\gamma)$ in $\mathfrak{z}_0$, $\mathfrak{p}_0$, $\mathfrak{k}_0$. We have

$$\mathfrak{z}_{\sigma,0}^\perp(\gamma) = \mathfrak{p}_{\sigma,0}^\perp(\gamma) \oplus \mathfrak{k}_{\sigma,0}^\perp(\gamma). \tag{4.3}$$

For $Y_0^\mathfrak{k} \in \mathfrak{k}_\sigma(\gamma)$, $\mathrm{ad}(Y_0^\mathfrak{k})$ preserves $\mathfrak{p}_\sigma(\gamma), \mathfrak{k}_\sigma(\gamma), \mathfrak{p}_{\sigma,0}^\perp(\gamma), \mathfrak{k}_{\sigma,0}^\perp(\gamma)$, and it is an antisymmetric endomorphism with respect to the scalar product.

As explain in [9, p. 105], the following function $A(Y_0^\mathfrak{k})$ in $Y_0^\mathfrak{k} \in \mathfrak{k}_\sigma(\gamma)$ has a natural square root that is analytic,

$$A(Y_0^\mathfrak{k}) = \frac{1}{\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{z}_{\sigma,0}^\perp(\gamma)}} \times \frac{\det\big(1-\exp(-\mathrm{i}\,\mathrm{ad}(Y_0^\mathfrak{k}))\,\mathrm{Ad}(k^{-1}\sigma)\big)|_{\mathfrak{k}_{\sigma,0}^\perp(\gamma)}}{\det\big(1-\exp(-\mathrm{i}\,\mathrm{ad}(Y_0^\mathfrak{k}))\,\mathrm{Ad}(k^{-1}\sigma)\big)|_{\mathfrak{p}_{\sigma,0}^\perp(\gamma)}}. \tag{4.4}$$

Its square root is denoted by

$$\left[\frac{1}{\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{z}_{\sigma,0}^\perp(\gamma)}} \cdot \frac{\det\big(1-\exp(-\mathrm{i}\,\mathrm{ad}(Y_0^\mathfrak{k}))\,\mathrm{Ad}(k^{-1}\sigma)\big)|_{\mathfrak{k}_{\sigma,0}^\perp(\gamma)}}{\det\big(1-\exp(-\mathrm{i}\,\mathrm{ad}(Y_0^\mathfrak{k}))\,\mathrm{Ad}(k^{-1}\sigma)\big)|_{\mathfrak{p}_{\sigma,0}^\perp(\gamma)}}\right]^{1/2}. \tag{4.5}$$

If $Y_0^\mathfrak{k} = 0$, this square root has the value $1/\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}_{\sigma,0}^\perp(\gamma)}$.

The following definition is a direct generalization of the function J_γ defined by [9, (5.5.5)], we often call them the J-functions.

Definition 4.1. — *Let $J_{\gamma\sigma}(Y_0^\mathfrak{k})$ be the analytic function of $Y_0^\mathfrak{k} \in \mathfrak{k}_\sigma(\gamma)$ given by*

$$J_{\gamma\sigma}(Y_0^\mathfrak{k}) = \frac{1}{\left|\det(1-\mathrm{Ad}(\gamma\sigma))|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{\widehat{A}\big(\mathrm{i}\,\mathrm{ad}(Y_0^\mathfrak{k})|_{\mathfrak{p}_\sigma(\gamma)}\big)}{\widehat{A}\big(\mathrm{i}\,\mathrm{ad}(Y_0^\mathfrak{k})|_{\mathfrak{k}_\sigma(\gamma)}\big)} \times\left[\frac{1}{\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{z}_{\sigma,0}^\perp(\gamma)}} \frac{\det\big(1-\exp(-\mathrm{i}\,\mathrm{ad}(Y_0^\mathfrak{k}))\,\mathrm{Ad}(k^{-1}\sigma)\big)|_{\mathfrak{k}_{\sigma,0}^\perp(\gamma)}}{\det\big(1-\exp(-\mathrm{i}\,\mathrm{ad}(Y_0^\mathfrak{k}))\,\mathrm{Ad}(k^{-1}\sigma)\big)|_{\mathfrak{p}_{\sigma,0}^\perp(\gamma)}}\right]^{1/2}. \tag{4.6}$$

By (4.6), there exist $c_{\gamma\sigma}$, $C_{\gamma\sigma} > 0$ such that,

$$\left|J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\right| \leqslant c_{\gamma\sigma} \exp\left(C_{\gamma\sigma}|Y_0^{\mathfrak{k}}|\right). \tag{4.7}$$

Remark 4.2. — If $t > 0$, if we replace B by B/t, the function $J_{\gamma\sigma}$ is unchanged.

4.2. A formula for the twisted orbital integrals for the heat kernel

Let $U\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$. If we identify $\mathfrak{g}$ to the vector space of left-invariant vector fields on G, then the enveloping algebra $U\mathfrak{g}$ is identified with the algebra of left-invariant differential operators on G.

Let $C^{\mathfrak{g}} \in U\mathfrak{g}$ be the Casimir element of G associated with the bilinear form B. If $e_1, \dots, e_{m+n}$ is a basis of $\mathfrak{g}$ and if $e_1^*, \dots, e_{m+n}^*$ is the dual basis of $\mathfrak{g}$ with respect to B, then

$$C^{\mathfrak{g}} = -\sum_{i=1}^{m+n} e_i^* e_i. \tag{4.8}$$

Also $C^{\mathfrak{g}}$ lies in the center of $U\mathfrak{g}$ and commutes with $\widetilde{G}$.

The scalar product $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ is given by $-B(\cdot, \theta \cdot)$. If $e_1, \dots, e_m$ is an orthonormal basis of $\mathfrak{p}$, and if $e_{m+1}, \dots, e_{m+n}$ is an orthonormal basis of $\mathfrak{k}$, by (4.8), we have

$$C^{\mathfrak{g}} = -\sum_{i=1}^{m} e_i^2 + \sum_{i=m+1}^{m+n} e_i^2. \tag{4.9}$$

Set

$$C^{\mathfrak{g},H} = -\sum_{i=1}^{m} e_i^2, \; C^{\mathfrak{k}} = \sum_{i=m+1}^{m+n} e_i^2. \tag{4.10}$$

Note that $C^{\mathfrak{k}} \in U\mathfrak{k}$ is just the Casimir element of K associated with $B|_{\mathfrak{k}}$.

By (4.8)–(4.10), we have

$$C^{\mathfrak{g}} = C^{\mathfrak{g},H} + C^{\mathfrak{k}}. \tag{4.11}$$

Let $F = G \times_K E$ be a homogeneous vector bundle on X defined from a unitary finite-dimensional K^σ-representation (E, ρ^E).

The operator $C^{\mathfrak{g}}$ acts on $C^\infty(X, F)$ via the identification (2.11). Let $C^{\mathfrak{g},X}$ denote the action of $C^{\mathfrak{g}}$ on $C^\infty(X, F)$, which commutes with G^σ.

Let $\Delta^{H,X}$ denote the Bochner Laplacian acting on $C^\infty(X,F)$. Then $C^{\mathfrak{g},H}$ descends to $-\Delta^{H,X}$. Let $C^{\mathfrak{k},E} \in \mathrm{End}(E)$ denote the action of $C^{\mathfrak{k}}$ on E given by

$$C^{\mathfrak{k},E} = \sum_{i=m+1}^{m+n} \rho^{E,2}(e_i). \tag{4.12}$$

Then $C^{\mathfrak{k},E}$ descends to an invariant parallel section $C^{\mathfrak{k},F}$ of $\mathrm{End}(F)$. By (4.11),

$$C^{\mathfrak{g},X} = -\Delta^{H,X} + C^{\mathfrak{k},F}. \tag{4.13}$$

Let $\kappa^{\mathfrak{g}} \in \Lambda^3(\mathfrak{g}^*)$ be such that if $a,b,c \in \mathfrak{g}$,

$$\kappa^{\mathfrak{g}}(a,b,c) = B([a,b],c). \tag{4.14}$$

The form $\kappa^{\mathfrak{g}}$ is invariant by the adjoint action of $G \rtimes \Sigma$. We can view $\kappa^{\mathfrak{g}}$ as a closed left and right invariant 3-form on G.

Let B^* be the bilinear form on $\Lambda^\bullet(\mathfrak{g}^*)$ induced by B. Let $C^{\mathfrak{k},\mathfrak{k}} \in \mathrm{End}(\mathfrak{k})$, $C^{\mathfrak{k},\mathfrak{p}} \in \mathrm{End}(\mathfrak{p})$ be the actions of $C^{\mathfrak{k}}$ on $\mathfrak{k}$, $\mathfrak{p}$ respectively via the adjoint actions of $\mathfrak{k}$ as in (4.12). By [9, (2.6.4), (2.6.11)],

$$B^*(\kappa^{\mathfrak{g}},\kappa^{\mathfrak{g}}) = \frac{1}{6}\sum_{i,j=1}^{m+n} B\left([e_i,e_j],[e_i^*,e_j^*]\right) = \frac{1}{2}\operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{1}{6}\operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]. \tag{4.15}$$

Definition 4.3. — *Let $\mathcal{L}^X$ be the operator acting on $C^\infty(X,F)$,*

$$\mathcal{L}^X = \frac{1}{2}C^{\mathfrak{g},X} + \frac{1}{8}B^*(\kappa^{\mathfrak{g}},\kappa^{\mathfrak{g}}). \tag{4.16}$$

Then $\mathcal{L}^X$ commutes with G^σ.

Let A be a self-adjoint element of $\mathrm{End}(E)$ which commutes with the action of K^σ. Then A descends to a self-adjoint parallel section of $\mathrm{End}(F)$ which commutes with G^σ.

Definition 4.4. — *Let $\mathcal{L}^X_A$ be the operator acting on $C^\infty(X,F)$,*

$$\mathcal{L}^X_A = \mathcal{L}^X + A. \tag{4.17}$$

It is clear that $\mathcal{L}^X_A$ is a Bochner-like Laplacian. For $t>0$, the heat operator $\exp(-t\mathcal{L}^X_A)$ has a smooth kernel $p^X_t(x,x')$ with respect to $\mathrm{d}x$ on X.

Proposition 4.5. — *For $t>0$, $p^X_t \in \mathcal{Q}^\sigma$.*

Proof. — This follows from [9, Proposition 4.4.2] and from the fact that $\mathcal{L}^X$ commutes with the action of σ. □

It follows from Subsection 3.2 and Proposition 4.5 that for $t>0$, the twisted orbital integral $\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_A^X)]$ is well-defined. Recall that $p=\dim\mathfrak{p}_\sigma(\gamma)$, $q=\dim\mathfrak{k}_\sigma(\gamma)$.

Theorem 4.6. — *For any $t>0$, the following identity holds:*

$$\mathrm{Tr}^{[\gamma\sigma]}\left[\exp(-t\mathcal{L}_A^X)\right]=\frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}} \tag{4.18}$$

$$\cdot\int_{\mathfrak{k}_\sigma(\gamma)} J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\,\mathrm{Tr}^E\left[\rho^E(k^{-1}\sigma)\exp(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}})-tA)\right]\mathrm{e}^{-|Y_0^{\mathfrak{k}}|^2/2t}\,\frac{\mathrm{d}Y_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.$$

Proof. — The proof of our theorem will be given in Section 6. □

As we explained in Subsection 3.2, our twisted orbital integral has an expression as an ordinary (un-twisted) orbital integral for a larger group G^σ (cf. (3.19)), whose Lie algebra is the semi-direct sum of $\mathfrak{g}$ and the Lie algebra of Σ^σ. One surprising point here is that in our formula (4.18), only the Lie subalgebras of $\mathfrak{g}$ appears, specially for the cases where σ is not of finite order. Indeed, in our setting, the twist σ plays a role of an equivariant action on the vector bundles, so that when we apply the local index techniques to prove (4.18), the Lie algebra of Σ^σ, as we will see, is not involved through the computations.

In Section 7, we will look into some geometric operators on X, such as Laplacians for spinors and Hodge Laplacians for flat vector bundles. They all can be written as $\mathcal{L}_A^X$ with suitable A's. Therefore, we can evaluate the corresponding equivariant heat traces via (3.45) and (4.18) for $Z=\Gamma\backslash X$.

4.3. The twisted orbital integrals of wave operators

Let $\Delta^{\mathfrak{z}_\sigma(\gamma)}$ be the standard Laplacian on $\mathfrak{z}_\sigma(\gamma)$ with respect to the scalar product on $\mathfrak{z}_\sigma(\gamma)$. For $t>0$, let $\exp(t\Delta^{\mathfrak{z}_\sigma(\gamma)}/2)$ be the corresponding heat operator. Let y, $Y_0^{\mathfrak{k}}$ denote the generic elements in $\mathfrak{p}_\sigma(\gamma)$, $\mathfrak{k}_\sigma(\gamma)$ respectively. Let $\mathrm{d}y\,\mathrm{d}Y_0^{\mathfrak{k}}$ be the Euclidean volume element of $\mathfrak{z}_\sigma(\gamma)$, and let $\exp(t\Delta^{\mathfrak{z}_\sigma(\gamma)}/2)((y,Y_0^{\mathfrak{k}}),(y',Y_0^{\mathfrak{k}\prime}))$ denote the smooth kernel of $\exp(t\Delta^{\mathfrak{z}_\sigma(\gamma)}/2)$ with respect to $\mathrm{d}y'\,\mathrm{d}Y_0^{\mathfrak{k}\prime}$.

Let $\delta_{y=a}$ be a distribution on $\mathfrak{z}_\sigma(\gamma)=\mathfrak{p}_\sigma(\gamma)\oplus\mathfrak{k}_\sigma(\gamma)$ associated with the affine subspace $\{y=a\}$. Then $J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\rho^E(k^{-1}\sigma)\exp(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}}))\delta_{y=a}$ is a distribution on $\mathfrak{z}_\sigma(\gamma)$ with values in $\mathrm{End}(E)$. Applying the heat operator $\exp(t\Delta^{\mathfrak{z}_\sigma(\gamma)}/2-tA)$ to this distribution, we get a smooth function on $\mathfrak{z}_\sigma(\gamma)$ with values in $\mathrm{End}(E)$. It can be evaluated at $0\in\mathfrak{z}_\sigma(\gamma)$. Then Theorem 4.6

can be rewritten as follows,

$$\begin{aligned}(4.19)\quad \operatorname{Tr}^{[\gamma\sigma]}\big[\exp(-t\mathcal{L}_A^X)\big] = \operatorname{Tr}^E\Big[\exp\Big(t\Delta^{\mathfrak{z}_\sigma(\gamma)}/2 - tA\Big)\\ \big[J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\rho^E(k^{-1}\sigma)\exp\big(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}})\big)\delta_{y=a}\big]\Big](0).\end{aligned}$$

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of $\mathbb{R}$, let $\mathcal{S}^{\mathrm{even}}(\mathbb{R})$ be the space of even functions in $\mathcal{S}(\mathbb{R})$. The Fourier transform of $\mu \in \mathcal{S}(\mathbb{R})$ is given by

$$(4.20)\qquad \widehat{\mu}(y) = \int_{\mathbb{R}} \mathrm{e}^{-2\mathrm{i}\pi yx}\,\mu(x)\mathrm{d}x.$$

Take $\mu \in \mathcal{S}^{\mathrm{even}}(\mathbb{R})$, then $\widehat{\mu} \in \mathcal{S}^{\mathrm{even}}(\mathbb{R})$. We now assume that there exists $C > 0$ such that for any $k \in \mathbb{N}$, there exists $c_k > 0$ such that

$$(4.21)\qquad |\widehat{\mu}^{(k)}(y)| \leqslant c_k \exp(-C|y|^2).$$

Then $\mu(\sqrt{\mathcal{L}^X + A})$ is a self-adjoint operator with a smooth kernel. We denote its smooth kernel by $\mu(\sqrt{\mathcal{L}^X + A})(x, x') \in \operatorname{Hom}(F_{x'}, F_x)$, $x, x' \in X$. As explained in [9, Section 6.2], we have

$$(4.22)\qquad \mu\Big(\sqrt{\mathcal{L}^X + A}\Big) \in \mathcal{Q}.$$

This is a consequence of the finite propagation speed for wave operators. We refer to [47, Section 4.4] for more details.

Since σ commutes with $\mathcal{L}^X + A$, we can get $\mu(\sqrt{\mathcal{L}^X + A}) \in \mathcal{Q}^\sigma$. Then the twisted orbital integral $\operatorname{Tr}^{[\gamma\sigma]}[\mu(\sqrt{\mathcal{L}^X + A})]$ is well-defined. Similarly, the kernel of $\mu(\sqrt{-\Delta^{\mathfrak{z}_\sigma(\gamma)}/2 + A})$ on $\mathfrak{z}_\sigma(\gamma)$ also has a Gaussian-like decay.

Using Theorem 4.6 and by (4.7), (4.19), (4.21), a modification of the proof to [9, Theorem 6.2.2], using essentially the denseness of $y^{2k}\,\mathrm{e}^{-ty^2/2}$, $k \in \mathbb{N}$ in $\mathcal{S}^{\mathrm{even}}(\mathbb{R})$, shows the following result.

Theorem 4.7. — *The following identity holds:*

$$\begin{aligned}(4.23)\quad \operatorname{Tr}^{[\gamma\sigma]}\Big[\mu\Big(\sqrt{\mathcal{L}^X + A}\Big)\Big] = \operatorname{Tr}^E\Big[\mu\Big(\sqrt{-\Delta^{\mathfrak{z}_\sigma(\gamma)}/2 + A}\Big)J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\\ \times \rho^E(k^{-1}\sigma)\exp\big(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}})\big)\delta_{y=a}\Big](0).\end{aligned}$$

Let $\operatorname{Tr}^{[\gamma\sigma]}[\cos(s\sqrt{\mathcal{L}^X + A})]$ be the even distribution on $s \in \mathbb{R}$ such that for any $\mu \in \mathcal{S}^{\mathrm{even}}(\mathbb{R})$ with $\widehat{\mu}$ having compact support,

$$(4.24)\qquad \operatorname{Tr}^{[\gamma\sigma]}\Big[\mu\Big(\sqrt{\mathcal{L}^X + A}\Big)\Big] = \int_{\mathbb{R}} \widehat{\mu}(s)\operatorname{Tr}^{[\gamma\sigma]}\Big[\cos\Big(s\sqrt{\mathcal{L}^X + A}\Big)\Big]\mathrm{d}s.$$

Let $P_\sigma^\perp(\gamma) \subset X$ be the image of $\mathfrak{p}_\sigma^\perp(\gamma)$ by the map $f \to p\,\mathrm{e}^f$. Put

$$(4.25)\qquad \Delta_X^{\gamma\sigma} = \{(x, \gamma\sigma(x)) \;:\; x \in P_\sigma^\perp(\gamma)\}.$$

Then $\Delta_X^{\gamma\sigma}$ is a submanifold of $X \times X$. We view $\mathbb{R} \times \Delta_X^{\gamma\sigma}$ as a distribution on $\mathbb{R} \times X \times X$. By analyzing the wave front sets for both $\cos(s\sqrt{\mathcal{L}^X + A})$ and $\mathbb{R} \times \Delta_X^{\gamma\sigma}$ ([25, Theorem 8.2.10], [26, Theorem 23.1.4]), we get that the distribution $\gamma\sigma \cos(s\sqrt{\mathcal{L}^X + A})(\mathbb{R} \times \Delta_X^{\gamma\sigma})$ is well-defined on $\mathbb{R} \times X \times X$. Using again the finite propagation speed of $\cos(s\sqrt{\mathcal{L}^X + A})$, we see that the push-forward of $\mathrm{Tr}^F[\gamma\sigma \cos(s\sqrt{\mathcal{L}^X + A})](\mathbb{R} \times \Delta_X^{\gamma\sigma})$ by the projection $\mathbb{R} \times X \times X \to \mathbb{R}$ is well-defined, which will be denoted by

$$\int_{\Delta_X^{\gamma\sigma}} \mathrm{Tr}^F \left[\gamma\sigma \cos\left(s\sqrt{\mathcal{L}^X + A}\right)\right]. \tag{4.26}$$

By (3.20), (4.24), (4.25), we have the identity of even distributions on $\mathbb{R}$,

$$\mathrm{Tr}^{[\gamma\sigma]}\left[\cos\left(s\sqrt{\mathcal{L}^X + A}\right)\right] = \int_{\Delta_X^{\gamma\sigma}} \mathrm{Tr}^F \left[\gamma\sigma \cos\left(s\sqrt{\mathcal{L}^X + A}\right)\right]. \tag{4.27}$$

The even distribution on $\mathbb{R}$,

$$\mathrm{Tr}^E\left[\cos\left(s\sqrt{-\Delta^{\mathfrak{z}_\sigma(\gamma)}/2+A}\right) J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\rho^E(k^{-1}\sigma)\exp\left(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}})\right)\delta_{y=a}\right](0) \tag{4.28}$$

is defined by

$$\begin{aligned} &\mathrm{Tr}^E\left[\mu\left(\sqrt{-\Delta^{\mathfrak{z}_\sigma(\gamma)}/2+A}\right)J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\rho^E(k^{-1}\sigma)\exp\left(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}})\right)\delta_{y=a}\right](0) \\ &\quad = \int_{\mathbb{R}} \widehat{\mu}(s)\,\mathrm{Tr}^E\left[\cos\left(2\pi s\sqrt{-\Delta^{\mathfrak{z}_\sigma(\gamma)}/2+A}\right)J_{\gamma\sigma}(Y_0^{\mathfrak{k}}) \right. \\ &\qquad\qquad \left. \rho^E(k^{-1}\sigma)\exp\left(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}})\right)\delta_{y=a}\right](0). \end{aligned} \tag{4.29}$$

Let $(a, \mathfrak{k}_\sigma(\gamma))$ denote the affine subspace of $\mathfrak{z}_\sigma(\gamma) = \mathfrak{p}_\sigma(\gamma) \oplus \mathfrak{k}_\sigma(\gamma)$. Set

$$H_\sigma^\gamma = \{0\} \times (a, \mathfrak{k}_\sigma(\gamma)) \subset \mathfrak{z}_\sigma(\gamma) \times \mathfrak{z}_\sigma(\gamma). \tag{4.30}$$

Then we have the tautological identification of even distributions on $\mathbb{R}$,

$$\begin{aligned} &\mathrm{Tr}^E\left[\cos\left(s\sqrt{-\Delta^{\mathfrak{z}_\sigma(\gamma)}/2+A}\right)J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\rho^E(k^{-1}\sigma)\exp\left(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}})\right)\delta_{y=a}\right](0) \\ &= \int_{H_\sigma^\gamma} \mathrm{Tr}^E\left[\cos\left(s\sqrt{-\Delta^{\mathfrak{z}_\sigma(\gamma)}/2+A}\right)J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\rho^E(k^{-1}\sigma)\exp\left(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}})\right)\right]. \end{aligned} \tag{4.31}$$

This is an analogue of (4.27).

Following the above constructions, we extend [9, Theorem 6.3.2] for the twisted orbital integrals, where the supports and singular supports of the above distributions are obtained as in [9, Proposition 6.3.1].

THEOREM 4.8. — *We have the identity of even distributions on* $\mathbb{R}$ *supported on* $\{s \in \mathbb{R} : |s| \geqslant \sqrt{2}|a|\}$ *with singular support included in* $\pm\sqrt{2}|a|$,

$$\int_{\Delta_X^{\gamma\sigma}} \mathrm{Tr}^F \left[\gamma\sigma \cos\left(s\sqrt{\mathcal{L}^X + A}\right)\right] \tag{4.32}$$
$$= \int_{H_\sigma^\gamma} \mathrm{Tr}^E \left[\cos\left(s\sqrt{-\Delta^{\mathfrak{z}_\sigma(\gamma)}/2 + A}\right) J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\rho^E(k^{-1}\sigma)\exp\left(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}})\right)\right].$$

4.4. Representation of K^σ and vanishing of twisted orbital integrals

In Subsections 3.1 and 3.2, for the twisted orbital integral, we always start with a K^σ-representation ρ^E. Now we study the irreducible representations of K^σ and show that only σ-stable irreducible representations of K give the non-vanishing twisted orbital integrals. Let $\mathrm{Irr}(\cdot)$ denote the set of equivalent classes of irreducible (complex) representations of a compact Lie group.

PROPOSITION 4.9. — *If* $(E, \rho^E) \in \mathrm{Irr}(K^\sigma)$ *and if the restriction of* (E, ρ^E) *to* K *is not irreducible, then for* $k \in K$, *we have*

$$\mathrm{Tr}^E[\rho^E(\sigma)\rho^E(k)] = 0. \tag{4.33}$$

Proof. — At first, we assume that K is semisimple. Let $\mathrm{Inn}(K)$ denote the inner automorphism group of K. The outer automorphism group of K is

$$\mathrm{Out}(K) = \mathrm{Aut}(K)/\,\mathrm{Inn}(K). \tag{4.34}$$

By fixing a maximal torus T of K and an associated positive root system R^+, $\mathrm{Out}(K)$ can be realized as a finite subgroup of $\mathrm{Aut}(K)$ whose elements preserve T and R^+ [16, Chapter VIII, §4.4 and Chapter IX, §4.10]. Moreover,

$$\mathrm{Aut}(K) = \mathrm{Inn}(K) \rtimes \mathrm{Out}(K). \tag{4.35}$$

Take $k_0 \in K$, $\tau \in \mathrm{Out}(K)$ such that for $k \in K$,

$$\sigma(k) = k_0\tau(k)k_0^{-1}. \tag{4.36}$$

Let K^τ be the subgroup of $K \rtimes \mathrm{Out}(K)$ generated by K and τ. We claim that there exists $c_\tau \in \mathbb{C}$ such that if set

$$\rho^{E,\prime}(\tau) = c_\tau\rho^E(k_0^{-1})\rho^E(\sigma), \quad \rho^{E,\prime}(k) = \rho^E(k), \tag{4.37}$$

then $(E,\rho^{E,\prime})$ is an irreducible representation of K^τ. Note that such number c_τ is not unique, it depends on the order of τ and the choice of k_0.

Indeed, set

$$A=\rho^E(k_0^{-1})\rho^E(\sigma)\in\mathrm{End}(E). \tag{4.38}$$

Let $N_0\geqslant 1$ be the order of τ in $\mathrm{Out}(K)$. Set

$$\widehat{k}=k_0\tau(k_0)\cdots\tau^{N_0-1}(k_0)\in K. \tag{4.39}$$

Then

$$\sigma(\widehat{k})=\widehat{k}\in K,\ \sigma^{N_0}=\mathrm{Ad}(\widehat{k})\in\mathrm{Inn}(K). \tag{4.40}$$

Also we have

$$A^{N_0}=\rho^E(\widehat{k}^{-1})\rho^E(\sigma^{N_0}). \tag{4.41}$$

We can verify directly that A^{N_0} commutes with K^σ. Since (E,ρ^E) is irreducible as K^σ-representation, then A^{N_0} is a non-zero scalar endomorphism of E, then we take $c_\tau\in\mathbb{C}^*$ such that $c_\tau^{N_0}A^{N_0}=\mathrm{Id}_E$.

We define $\rho^{E,\prime}$ as in (4.37). Then for $k\in K$,

$$\rho^{E,\prime}(\tau)\rho^{E,\prime}(k)\rho^{E,\prime}(\tau^{-1})=\rho^{E,\prime}(\tau(k)). \tag{4.42}$$

Therefore, $(E,\rho^{E,\prime})$ become an irreducible representation of K^τ.

By (4.37), for any $k\in K$, we have

$$\begin{aligned}\mathrm{Tr}^E[\rho^{E,\prime}(\tau)\rho^{E,\prime}(k)]&=\mathrm{Tr}^E[c_\tau\rho^E(k_0^{-1})\rho^E(\sigma)\rho^E(k)]\\&=c_\tau\,\mathrm{Tr}^E[\rho^E(\sigma)\rho^E(kk_0^{-1})].\end{aligned} \tag{4.43}$$

Note that $c_\tau\neq 0$. Then for proving (4.33), it is equivalent to prove that for all $k\in K$,

$$\mathrm{Tr}^E[\rho^{E,\prime}(\tau)\rho^{E,\prime}(k)]=0. \tag{4.44}$$

In the sequel, we prove (4.44). Let P_{++} be the dominant weights for the pair (K,T) with respect to R^+. Then τ acts on P_{++}. If $\lambda\in P_{++}$, let $V_\lambda\in\mathrm{Irr}(K)$ denote the one with the highest weight λ.

Now we take a dominant weight $\lambda\in P_{++}$ such that V_λ embeds into (E,ρ^E) as a K-subrepresentation. Let $\{\tau^i(\lambda)\}_{i=0}^{d-1}\subset P_{++}$ be the orbit of λ under the action of τ. Note that $d\geqslant 1$ is the length of the orbit and $d\,|\,N_0$. By the description of all the irreducible representations of non-connected compact Lie groups in [20, Corollary 4.13.2 and Proposition 4.13.3], we get that the representation $(E,\rho^{E,\prime})$ restricting on K is of the form

$$\bigoplus_{i=0}^{d-1}V_{\tau^i(\lambda)}. \tag{4.45}$$

Moreover, the action $\rho^{E,\prime}(\tau)$ on E sends the component $V_{\tau^i(\lambda)}$ to $V_{\tau^{i+1}(\lambda)}$. As a consequence, we get (4.44).

If K is not semisimple (but always reductive), let Z^0_K be the identity component of the center of K, and let K_{ss} be the analytic subgroup of K associated with $\mathfrak{k}_{\mathrm{ss}} = [\mathfrak{k}, \mathfrak{k}]$. Then $Z^0_K \times K_{\mathrm{ss}}$ is a finite cover of K. Note that Z^0_K is a torus, the action of σ on it is of finite order. Then if we proceed as in the above for K_{ss}, we can still apply [20, Corollary 4.13.2 and Proposition 4.13.3] to get (4.45) and then (4.44). This completes the proof of our proposition. □

By our formula in Theorem 4.6, the integrand contains a ρ^E-trace term $\mathrm{Tr}^E[\rho^E(\sigma)\cdots]$. Then we get the following result.

Corollary 4.10. — *Let F be the Hermitian vector bundle on X defined from $(E, \rho^E) \in \mathrm{Irr}(K^\sigma)$ which is not irreducible as K-representation. Then for semisimple $\gamma\sigma$ as before, and for $t > 0$,*

$$\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_A)] = 0. \tag{4.46}$$

Moreover, if $\mu \in \mathcal{S}^{\mathrm{even}}(\mathbb{R})$ is such that (4.21) holds, then

$$\mathrm{Tr}^{[\gamma\sigma]}\left[\mu\Big(\sqrt{\mathcal{L}^X + A}\Big)\right] = 0. \tag{4.47}$$

Remark 4.11. — In [20, Section 4.13], a Weyl character formula for the non-connected compact Lie group (such as K^τ) was established. Then, via (4.37), the trace term $\mathrm{Tr}^E[\rho^E(\sigma)\cdots]$ in (4.18) can be written in terms of λ and the root data associated with (K, T). In Subsection 7.2, we use this observation to evaluate the twisted orbital integrals more explicitly in the geometric context.

The proof of Proposition 4.9 also gives a correspondence between $\mathrm{Irr}(K^\sigma)$ and τ-orbits in P_{++}. For simplicity, we assume K to be semisimple. Note that τ generates a finite group $\langle\tau\rangle$ in $\mathrm{Out}(K)$. The set $\mathrm{Irr}(\langle\tau\rangle)$ can be viewed as a finite abelian group ($\simeq \mathbb{Z}/N_0\mathbb{Z}$), it acts on $\mathrm{Irr}(K^\tau)$ by tensor product of representations. By [20, Corollary 4.13.2 and Proposition 4.13.3], we have the canonical bijection,

$$\mathrm{Irr}(\langle\tau\rangle)\backslash \mathrm{Irr}(K^\tau) \simeq \langle\tau\rangle\backslash P_{++}. \tag{4.48}$$

Similarly, $\mathrm{Irr}(\Sigma^\sigma)$ acts on $\mathrm{Irr}(K^\sigma)$. Then the construction given by (4.37) implies an injective map

$$\mathrm{Irr}(\Sigma^\sigma)\backslash \mathrm{Irr}(K^\sigma) \to \mathrm{Irr}(\langle\tau\rangle)\backslash \mathrm{Irr}(K^\tau). \tag{4.49}$$

Now we explain that the map in (4.49) is also a bijection. By (4.48), we consider a τ-orbit in P_{++}, and let λ be one element in this orbit. Let

$\mathrm{Ind}_K^{K^\sigma}(V_\lambda)$ be the induced K^σ-representation, and let (E, ρ^E) be a K^σ-irreducible component of it, which contains a V_λ-component when restricting to K. Then, by the arguments as in (4.37)–(4.45), we get the representation $(E, \rho^E) \in \mathrm{Irr}(K^\sigma)$ corresponds exactly to the τ-orbit of λ in P_{++}. Therefore, the map in (4.49) is surjective, then a bijection.

PROPOSITION 4.12. — *If (E, ρ^E) is a finite dimensional unitary K-representation, then it can extend to an irreducible representation of K^σ if and only if the highest weights of its K-irreducible components form exactly one τ-orbit in P_{++}. Therefore, an irreducible K^σ-representation is also K-irreducible if and only if its highest weight is fixed by τ.*

4.5. Examples from cyclic base change theory

The twisted orbital integral plays an important role in the cyclic base change theory, where σ is of finite order. The typical examples are as follows,

- a connected semisimple complex linear Lie group $G_{\mathbb{C}}$, where σ is taken to be the conjugation of a matrix and its fixed point set is just the the real matrix subgroup $G_{\mathbb{R}}$;
- the product case where $G = G_0^\ell$ is given as ℓ-copies of a connected real semisimple Lie group G_0 and σ is given as the cyclic permutation of the copies. The simplest case is $\ell = 2$.

In this subsection, we focus on such examples and explain how to make use of our formula in Theorem 4.6. In particular, we show via elementary computations how the twisted orbital integrals (for $G_{\mathbb{C}}$ or G_0^ℓ) relate to the ordinary orbital integrals (for $G_{\mathbb{R}}$ or G_0). Note that we have no any regularity condition on the semisimple element $\gamma\sigma$.

Example 4.13 (Complex semisimple Lie group and matrix conjugation). Let $N \in \mathbb{N}$ be large integer. Let $G = G_{\mathbb{C}} \subset \mathrm{GL}(N, \mathbb{C})$ be a connected and simply connected semisimple linear algebraic group which is invariant under transpose and conjugation. Then the Cartan involution is given as $\theta(A) = (\overline{A}^T)^{-1}$, where $(\cdot)^T$ denotes the matrix transpose.

We also view $G_{\mathbb{C}}$ as a real semisimple Lie group with (real) Lie algebra $\mathfrak{g}$, the bilinear form B on $\mathfrak{g}$ is taken to be the real trace form, which is equal to the real part of the complex Killing form on $\mathfrak{g}_{\mathbb{C}}$ up a positive multiple. Let $\sigma \in \mathrm{Aut}(G_{\mathbb{C}})$ be such that $\sigma(A) = \overline{A}$. Then its fixed points are exactly the real points of $G_{\mathbb{C}}$, denoted by $G_{\mathbb{R}}$, the subgroup of real matrices in $G_{\mathbb{C}}$. Alternatively speaking, $G_{\mathbb{R}}$ is a split real form of $G_{\mathbb{C}}$, and $G_{\mathbb{C}}$ is the complexification of $G_{\mathbb{R}}$. We will put $X_{\mathbb{C}} = G_{\mathbb{C}}/K$, $X_{\mathbb{R}} = G_{\mathbb{R}}/K_{\mathbb{R}}$.

Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{p}_{\mathbb{R}} \oplus \mathfrak{k}_{\mathbb{R}}$ denote the Cartan decomposition of Lie algebra of $G_{\mathbb{R}}$, and let $K_{\mathbb{R}} \subset G_{\mathbb{R}}$ be the maximal compact subgroup corresponding $\mathfrak{k}_{\mathbb{R}}$. Then

$$\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \oplus \mathrm{i}\mathfrak{g}_{\mathbb{R}}, \tag{4.50}$$

and the Cartan decomposition is given by

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k},\ \mathfrak{p} = \mathfrak{p}_{\mathbb{R}} \oplus \mathrm{i}\mathfrak{k}_{\mathbb{R}},\ \mathfrak{k} = \mathfrak{k}_{\mathbb{R}} \oplus \mathrm{i}\mathfrak{p}_{\mathbb{R}}. \tag{4.51}$$

Then the maximal compact subgroup K (with Lie algebra $\mathfrak{k}$) of $G_{\mathbb{C}}$ is just the compact real form of $G_{\mathbb{C}}$, and also the compact form of $G_{\mathbb{R}}$. Moreover, it is simply connected. For $\gamma \in G_{\mathbb{C}}$, if $\gamma\sigma$ is semisimple if and only if $\gamma\sigma(\gamma) \in G_{\mathbb{C}}$ is semisimple (cf. [17, Lemme 2.2]). Here, we consider the elliptic element σ itself, for which the associated twisted orbital integral $\mathrm{Tr}^{[\sigma]}[\cdot]$ has been studied vastly (cf. [18, §8], [15], [4], etc). In previous notation, we have $Z_\sigma(1) = G_{\mathbb{R}}$, $K_\sigma(1) = K_{\mathbb{R}}$. Set $p = \dim \mathfrak{p}_{\mathbb{R}}$, $q = \dim \mathfrak{k}_{\mathbb{R}}$.

We consider the following representations of $G_{\mathbb{C}}$. Let (E_0, ρ_0) be a finite dimensional holomorphic representation of $G_{\mathbb{C}}$, the unitary trick implies that the restriction of ρ_0 to K or $G_{\mathbb{R}}$ determines uniquely ρ_0. Let $(E_0^\sigma := E_0, \rho_0^\sigma)$ be the representation of $G_{\mathbb{C}}$ twisted by σ, i.e., $\rho_0^\sigma(g) = \rho_0(\sigma(g))$, $g \in G_{\mathbb{C}}$. For $v_1, v_2 \in E_0$, set

$$\rho^E(\sigma)(v_1 \otimes v_2) = v_2 \otimes v_1 \in E_0 \otimes E_0^\sigma. \tag{4.52}$$

This way, we get a representation $(E, \rho^E) := (E_0 \otimes E_0^\sigma, \rho_0 \otimes \rho_0^\sigma)$ of $(G_{\mathbb{C}})^\sigma = G_{\mathbb{C}} \rtimes \{1, \sigma\}$. Taking a K-invariant Hermitian metric on E_0, we make (E, ρ^E) as a unitary representation of $K^\sigma = K \rtimes \{1, \sigma\}$. We consider the Laplacian $\mathcal{L}^{X_{\mathbb{C}}, F}$ acting on $C^\infty(X_{\mathbb{C}}, F = G_{\mathbb{C}} \times_K E)$ defined in (4.16).

For $Y \in \mathfrak{k}_{\mathbb{R}}$, the J-function $J_1^{G_{\mathbb{R}}}$ for the identity element $1 \in G_{\mathbb{R}}$ is

$$J_1^{G_{\mathbb{R}}}(Y) = \frac{\widehat{A}\big(\mathrm{i}\,\mathrm{ad}(Y)|_{\mathfrak{p}_{\mathbb{R}}}\big)}{\widehat{A}\big(\mathrm{i}\,\mathrm{ad}(Y)|_{\mathfrak{k}_{\mathbb{R}}}\big)}. \tag{4.53}$$

Note that one should not confuse the imaginary unit i appearing in the J-functions with the one in the Lie algebra $\mathfrak{g}$.

An elementary computation shows that as a function in $Y \in \mathfrak{k}_{\mathbb{R}}$,

$$\widehat{A}\big(\mathrm{i}\,\mathrm{ad}(Y)|_{\mathfrak{p}_{\mathbb{R}}}\big) \left[\frac{1}{\det(1 + \mathrm{e}^{-\mathrm{i}\,\mathrm{ad}(Y)})|_{\mathfrak{p}_{\mathbb{R}}}}\right]^{1/2} = \frac{1}{2^{p/2}} \widehat{A}\big(\mathrm{i}\,\mathrm{ad}(2Y)|_{\mathfrak{p}_{\mathbb{R}}}\big). \tag{4.54}$$

Similar for $\widehat{A}\big(\mathrm{i}\,\mathrm{ad}(Y)|_{\mathfrak{k}_{\mathbb{R}}}\big)$. The twist σ acts on $\mathrm{i}\mathfrak{p}_{\mathbb{R}} \oplus \mathrm{i}\mathfrak{k}_{\mathbb{R}}$ as -1. Let J_σ be the J-function associated with σ and $G_{\mathbb{C}}$, then for $Y \in \mathfrak{k}_{\mathbb{R}} = \mathfrak{k}_\sigma(1)$,

$$J_\sigma(Y) = \frac{1}{2^p} J_1^{G_{\mathbb{R}}}(2Y) \left[\frac{\det(1 + \mathrm{e}^{-\mathrm{i}\,\mathrm{ad}(Y)})|_{\mathfrak{p}_{\mathbb{R}}}}{\det(1 + \mathrm{e}^{-\mathrm{i}\,\mathrm{ad}(Y)})|_{\mathfrak{k}_{\mathbb{R}}}}\right]. \tag{4.55}$$

For the trace of ρ^E, we have, for $Y \in \mathfrak{k}_{\mathbb{R}}$,

$$\operatorname{Tr}^E\left[\rho^E(\sigma)\exp(-\mathrm{i}\rho^E(Y))\right] = \operatorname{Tr}^{E_0}\left[\exp(-\mathrm{i}\rho^{E_0}(2Y))\right]. \tag{4.56}$$

By (4.18) in our theorem, we have, for $t > 0$,

$$\operatorname{Tr}^{[\sigma]}[\exp(-t\mathcal{L}^{X_{\mathbb{C}},F})] = \frac{1}{(8\pi t)^{p/2}} \int_{\mathfrak{k}_{\mathbb{R}}} J_1^{G_{\mathbb{R}}}(2Y) \times \left[\frac{\det(1+\mathrm{e}^{-\mathrm{i}\,\mathrm{ad}(Y)})|_{\mathfrak{p}_{\mathbb{R}}}}{\det(1+\mathrm{e}^{-\mathrm{i}\,\mathrm{ad}(Y)})|_{\mathfrak{k}_{\mathbb{R}}}}\right] \operatorname{Tr}^{E_0}\left[\mathrm{e}^{-\mathrm{i}\rho^{E_0}(2Y)}\right] \mathrm{e}^{-\frac{|Y|^2}{2t}} \frac{\mathrm{d}Y}{(2\pi t)^{q/2}}. \tag{4.57}$$

As we will see in Section 7, after twisting (E, ρ^E) with the graded (virtual) K^σ-representations $\rho^{\Lambda^\bullet(\mathfrak{p}^*)}$ on $\sum_j (-1)^j \Lambda^j(\mathfrak{p}^*)$ or $\sum_j (-1)^j j \Lambda^j(\mathfrak{p}^*)$, the denominator $\det(1+\mathrm{e}^{-\mathrm{i}\,\mathrm{ad}(Y)})|_{\mathfrak{k}_{\mathbb{R}}}$ can be canceled out properly. Such constructions, in geometric setting, appear in the evaluations of Lefschetz numbers or equivariant real analytic torsions. We will use $\operatorname{Tr_s}^{[\bullet]}[\cdots]$ with the subscript s to denote the (twisted) orbital integrals which take the supertrace of the endomorphisms of the $\mathbb{Z}_2$-graded vector bundles.

If $\mathfrak{t}_{\mathbb{R}}$ is a Cartan subalgebra of $\mathfrak{k}_{\mathbb{R}}$, put

$$\mathfrak{b}_{\mathbb{R}} = \{f \in \mathfrak{p}_{\mathbb{R}} \ : \ [f, v] = 0, \text{ for all } v \in \mathfrak{t}_{\mathbb{R}}\}.$$

Then $\mathfrak{t}_{\mathbb{R}} \oplus \mathfrak{b}_{\mathbb{R}}$ is Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$ ([28, p. 129]), and the fundamental rank $\delta(G_{\mathbb{R}})$ is defined as $\dim_{\mathbb{R}} \mathfrak{b}_{\mathbb{R}}$. Let $N^{\Lambda^\bullet(\mathfrak{p}^*)}$ denote the number operator on $\Lambda^\bullet(\mathfrak{p}^*)$ which acts on $\Lambda^j(\mathfrak{p}^*)$ as multiplication by j. Then we have the following identities for $Y \in \mathfrak{k}_{\mathbb{R}}$,

$$\left[\frac{\det(1+\mathrm{e}^{-\mathrm{i}\,\mathrm{ad}(Y)})|_{\mathfrak{p}_{\mathbb{R}}}}{\det(1+\mathrm{e}^{-\mathrm{i}\,\mathrm{ad}(Y)})|_{\mathfrak{k}_{\mathbb{R}}}}\right] \operatorname{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)}\left[\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(\sigma)\,\mathrm{e}^{-\mathrm{i}\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(Y)}\right] = \begin{cases} \operatorname{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*_{\mathbb{R}})}\left[\mathrm{e}^{-\mathrm{i}\rho^{\Lambda^\bullet(\mathfrak{p}^*_{\mathbb{R}})}(2Y)}\right] & \text{if } \delta(G_{\mathbb{R}}) = 0; \\ 0 & \text{if } \delta(G_{\mathbb{R}}) \geqslant 1, \end{cases} \tag{4.58}$$

and

$$\left[\frac{\det(1+\mathrm{e}^{-\mathrm{i}\,\mathrm{ad}(Y)})|_{\mathfrak{p}_{\mathbb{R}}}}{\det(1+\mathrm{e}^{-\mathrm{i}\,\mathrm{ad}(Y)})|_{\mathfrak{k}_{\mathbb{R}}}}\right] \operatorname{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)}\left[N^{\Lambda^\bullet(\mathfrak{p}^*)}\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(\sigma)\,\mathrm{e}^{-\mathrm{i}\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(Y)}\right] = \begin{cases} \left(\frac{p+q}{2}\right) \operatorname{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*_{\mathbb{R}})}\left[\mathrm{e}^{-\mathrm{i}\rho^{\Lambda^\bullet(\mathfrak{p}^*_{\mathbb{R}})}(2Y)}\right] & \text{if } \delta(G_{\mathbb{R}}) = 0; \\ 2\operatorname{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*_{\mathbb{R}})}\left[N^{\Lambda^\bullet(\mathfrak{p}^*_{\mathbb{R}})}\,\mathrm{e}^{-\mathrm{i}\rho^{\Lambda^\bullet(\mathfrak{p}^*_{\mathbb{R}})}(2Y)}\right] & \text{if } \delta(G_{\mathbb{R}}) = 1; \\ 0 & \text{if } \delta(G_{\mathbb{R}}) \geqslant 2. \end{cases} \tag{4.59}$$

We briefly explain how to obtain the above identities. Note that if g is an isometry of a finite dimensional Euclidean space V, then

$$\begin{aligned} \mathrm{Tr_s}^{\Lambda^\bullet(V^*)}[g] &= \det(1-g^{-1})|_V, \\ \mathrm{Tr_s}^{\Lambda^\bullet(V^*)}\left[N^{\Lambda^\bullet(V^*)} g\right] &= \frac{\partial}{\partial s}|_{s=0} \det(1-g^{-1}\,\mathrm{e}^{s})|_V. \end{aligned} \tag{4.60}$$

Moreover, if V is even-dimensional and g preserves the orientation, or if V is odd-dimensional and g reverses the orientation, then

$$\mathrm{Tr_s}^{\Lambda^\bullet(V^*)}\left[\left(N^{\Lambda^\bullet(V^*)} - \frac{\dim V}{2}\right) g\right] = 0. \tag{4.61}$$

Due to the invariance by adjoint action of $K_{\mathbb{R}}$, we only need to prove (4.58), (4.59) for $Y \in \mathfrak{t}_{\mathbb{R}}$. In this case, $\mathrm{ad}(Y)$ acts $\mathfrak{b}_{\mathbb{R}}$ as zero. Note that $\mathfrak{p} = \mathfrak{p}_{\mathbb{R}} \oplus \mathrm{i}\mathfrak{k}_{\mathbb{R}}$, then the first part of (4.58) follows directly from the first identity in (4.60). Using further (4.61), we get the first case ($\delta(G_{\mathbb{R}}) = 0$) in (4.59). The case where $\delta(G_{\mathbb{R}}) = \dim_{\mathbb{R}} \mathfrak{b}_{\mathbb{R}} \geqslant 2$ follows from the second identity in (4.60). Finally, when $\delta(G_{\mathbb{R}}) = 1$, $\mathfrak{b}_{\mathbb{R}}$ is a real line, then, by taking the orthogonal splitting $\mathfrak{p}_{\mathbb{R}} = \mathfrak{b}_{\mathbb{R}} \oplus \mathfrak{b}_{\mathbb{R}}^{\perp}$, the corresponding result in (4.59) follows from

$$\mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}_{\mathbb{R}}^*)}\left[N^{\Lambda^\bullet(\mathfrak{p}_{\mathbb{R}}^*)}\,\mathrm{e}^{-\mathrm{i}\rho^{\Lambda^\bullet(\mathfrak{p}_{\mathbb{R}}^*)}(Y)}\right] = -\det(1-\mathrm{e}^{-\mathrm{i}\,\mathrm{ad}(Y)})|_{\mathfrak{b}_{\mathbb{R}}^{\perp}}. \tag{4.62}$$

As a consequence, if $\delta(G_{\mathbb{R}}) = 0$,

$$\begin{aligned} &\mathrm{Tr_s}^{[\sigma]}\left[\exp(-t\mathcal{L}^{X_{\mathbb{C}},\Lambda^\bullet(T^*X_{\mathbb{C}})\otimes F})\right] \\ &\qquad = \mathrm{Tr_s}^{[1]}\left[\exp(-4t\mathcal{L}^{X_{\mathbb{R}},\Lambda^\bullet(T^*X_{\mathbb{R}})\otimes F_0})\right], \end{aligned} \tag{4.63}$$

and if $\delta(G_{\mathbb{R}}) = 1$,

$$\begin{aligned} &\mathrm{Tr_s}^{[\sigma]}\left[N^{\Lambda^\bullet(T^*X_{\mathbb{C}})}\exp(-t\mathcal{L}^{X_{\mathbb{C}},\Lambda^\bullet(T^*X_{\mathbb{C}})\otimes F})\right] \\ &\qquad = 2\,\mathrm{Tr_s}^{[1]}\left[N^{\Lambda^\bullet(T^*X_{\mathbb{R}})}\exp(-4t\mathcal{L}^{X_{\mathbb{R}},\Lambda^\bullet(T^*X_{\mathbb{R}})\otimes F_0})\right]. \end{aligned} \tag{4.64}$$

The other identities in (4.59) will imply the vanishing of Lefschetz numbers or equivariant analytic torsions, we refer to Section 7 and also [36, Theorem 3.3.2] for such results.

Example 4.14 (Product case). — Let $(G_0, K_0, \theta_0, B_0)$ be a connected real reductive Lie group. Put

$$(G, K, \theta) = (G_0, K_0, \theta_0) \times (G_0, K_0, \theta_0). \tag{4.65}$$

Let $\mathfrak{g}_0 = \mathfrak{k}_{G_0} \oplus \mathfrak{p}_{G_0}$ denote the Cartan decomposition of the Lie algebra of G_0. Then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0,\ \mathfrak{k} = \mathfrak{k}_{G_0} \oplus \mathfrak{k}_{G_0}, \mathfrak{p} = \mathfrak{p}_{G_0} \oplus \mathfrak{p}_{G_0}. \tag{4.66}$$

We define the bilinear form on $\mathfrak{g}$ by

$$B = B_0 \oplus B_0. \tag{4.67}$$

The symmetric space X is identify with $X_0 \times X_0$, where $X_0 = G_0/K_0$.

The twist σ is defined as follows, for $(g_1, g_2) \in G = G_0 \times G_0$,

$$\sigma(g_1, g_2) = (g_2, g_1). \tag{4.68}$$

The fixed point set of σ, i.e. the σ-twisted centralizer $Z_\sigma(1)$ of $1 \in G$, is exactly the diagonal of the product $G_0 \times G_0$. Then $Z_\sigma(1) \simeq G_0$ canonically, and the induced Cartan involution on $Z_\sigma(1)$ from θ is just θ_0. By (4.67), the bilinear form B restricting to $\mathfrak{z}_\sigma(1) \simeq \mathfrak{g}_0$ coincides with $2B_0$.

Let (E_0, ρ_0) be a unitary representation of K_0. Set $(E, \rho^E) = (E_0, \rho_0) \otimes (E_0, \rho_0)$, a unitary representation of K. For $v_1, v_2 \in E_0$, set $\rho^E(\sigma)(v_1 \otimes v_2) = v_2 \otimes v_1$. Then (E, ρ^E) extends as a representation of K^σ. We define the vector bundles F, F_0 on X, X_0 respectively. Let $\mathcal{L}^{X,F}$, $\mathcal{L}^{X_0,F_0}$ denote the operators as in (4.16) acting on $C^\infty(X, F)$, $C^\infty(X_0, F_0)$ respectively. In particular, we have

$$\mathcal{L}^{X,F} = \mathcal{L}^{X_0,F_0} \otimes 1 + 1 \otimes \mathcal{L}^{X_0,F_0}. \tag{4.69}$$

In [32, §8], under the above setting, Langlands deduced an identity between the σ-twisted orbitals integrals and the ordinary orbital integrals, where the matching functions are given via convolution. We specialize his result in our simple example here. For $\gamma_1, \gamma_2 \in G_0$, take $\gamma = (\gamma_1, \gamma_2) \in G$ such that $\gamma\sigma$ semisimple. We may assume that

$$\gamma_1 = \mathrm{e}^{a_1}\, k_1^{-1},\ \gamma_2 = \mathrm{e}^{a_2}\, k_2^{-1}\ a_1, a_2 \in \mathfrak{p}_{G_0}, k_1, k_2 \in K_0, \tag{4.70}$$

and by Theorem 2.7, $\mathrm{Ad}(k_1^{-1})a_2 = a_1, \mathrm{Ad}(k_2^{-1})a_1 = a_2$. The norm of γ is defined as $N\gamma = \gamma_1\gamma_2 \in G_0$, which has the form

$$\gamma_1\gamma_2 = \mathrm{e}^{a}\, k^{-1}, a = 2a_1, k = k_2k_1, \mathrm{Ad}(k)a = a. \tag{4.71}$$

Then $\gamma_1\gamma_2$ is a semisimple element in G_0.

Let $Z_0(N\gamma)$ be the centralizer of $N\gamma$ in G_0 with Lie algebra $\mathfrak{z}_{G_0}(N\gamma) = \mathfrak{p}_{G_0}(N\gamma) \oplus \mathfrak{k}_{G_0}(N\gamma) \subset \mathfrak{g}_0$. Then by (2.51), we have

$$Z_\sigma(\gamma) = \{(g, k_1 g k_1^{-1}) \in G \ :\ g \in Z_0(N\gamma)\} \simeq Z_0(N\gamma). \tag{4.72}$$

The diffeomorphism $(g_1, g_2) \in G \mapsto (g_1, g_2^{-1}\gamma_2 g_1) \in G$ induces an identification

$$Z_\sigma(\gamma)\backslash G \simeq (Z_0(N\gamma)\backslash G_0) \times G_0. \tag{4.73}$$

The result in [32, §8], as a consequence of (4.73), says that for $t > 0$,

$$\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^{X,F})] = \frac{1}{2^{p/2}}\, \mathrm{Tr}^{[N\gamma]}[\exp(-2t\mathcal{L}^{X_0,F_0})], \tag{4.74}$$

where the right-hand side is the ordinary orbital integrals for (G_0, B_0), and the factor $2^{p/2}$ comes from the volume conventions with $B|_{\mathfrak{z}_\sigma(\gamma)} = 2B_0|_{\mathfrak{z}_{G_0}(N\gamma)}$.

Now we explain how our formula (4.18) is compatible with (4.74). We start with the J-function $J_{\gamma\sigma}$. Set $p = \dim \mathfrak{p}_{G_0}(N\gamma), q = \dim \mathfrak{k}_{G_0}(N\gamma)$. For $Y \in \mathfrak{k}_{G_0}(N\gamma)$, $(Y, \mathrm{Ad}(k_1)Y) \in \mathfrak{k}_\sigma(\gamma)$. In this case, $\mathfrak{z}_0 = \mathfrak{z}((a_1, a_2)) = \mathfrak{z}_{G_0,0} \oplus \mathrm{Ad}(k_1)\mathfrak{z}_{G_0,0}$, where $\mathfrak{z}_{G_0,0} = \mathfrak{z}_{G_0}(a) \subset \mathfrak{g}_0$. Then we have

$$\mathfrak{k}^\perp_{\sigma,0}(\gamma) \simeq \big(\mathfrak{k}^\perp_{G_0,0}(N\gamma), \mathrm{Ad}(k_1)\mathfrak{k}^\perp_{G_0,0}(N\gamma)\big) \oplus \mathfrak{k}_{G_0}(N\gamma). \tag{4.75}$$

As a consequence, we get

$$\begin{aligned}\det\big(1 - \exp(-\mathrm{i}\,\mathrm{ad}(Y, \mathrm{Ad}(k_1)Y))\,\mathrm{Ad}((k_1,k_2)^{-1}\sigma)\big)|_{\mathfrak{k}^\perp_{\sigma,0}(\gamma)} \\ = \det\big(1 - \exp(-\mathrm{i}\,\mathrm{ad}(2Y))\,\mathrm{Ad}(k^{-1})\big)|_{\mathfrak{k}^\perp_{G_0,0}(N\gamma)} \\ \cdot \det\big(1 + \exp(-\mathrm{i}\,\mathrm{ad}(Y))\big)|_{\mathfrak{k}_{G_0}(N\gamma)}.\end{aligned} \tag{4.76}$$

Similar computations hold for $\mathfrak{p}^\perp_{\sigma,0}(\gamma)$ and $\mathfrak{z}^\perp_{\sigma,0}(\gamma)$. Then by (4.6) and (4.54),

$$J_{\gamma\sigma}(Y, \mathrm{Ad}(k_1)Y) = \frac{1}{2^p} J^{G_0}_{N\gamma}(2Y), \tag{4.77}$$

where $J^{G_0}_{N\gamma}$ is the corresponding J-function defined with $N\gamma$ and G_0.

Moreover, a direct computation shows,

$$\begin{aligned}\mathrm{Tr}^E\left[\rho^E((k_1,k_2)^{-1}\sigma)\exp\big(-\mathrm{i}\rho^E(Y, \mathrm{Ad}(k_1)Y)\big)\right] \\ = \mathrm{Tr}^{E_0}\left[\rho_0(k^{-1})\exp\left(-\mathrm{i}2\rho_0(Y)\right)\right].\end{aligned} \tag{4.78}$$

Note that $|(Y, \mathrm{Ad}(k_1)Y)|^2_B = 2|Y|^2_{B_0}$, where the subscripts indicate the corresponding norms. Then by (4.18), we get

$$\begin{aligned}\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^{X,F})] = \frac{1}{2^{p/2}} \frac{\exp(-|a|^2_{B_0}/4t)}{(4\pi t)^{p/2}} \\ \cdot \int_{\mathfrak{k}_{G_0}(N\gamma)} J^{G_0}_{N\gamma}(2Y)\,\mathrm{Tr}^{E_0}\left[\rho_0(k^{-1})\exp(-\mathrm{i}\rho_0(2Y))\right] \mathrm{e}^{-2|Y|^2_{B_0}/2t} \frac{2^q |\mathrm{d}Y|_{B_0}}{(4\pi t)^{q/2}}.\end{aligned} \tag{4.79}$$

After the coordinate change $2Y \to Y$ in the above integral, we get exactly $\frac{1}{2^{p/2}}\mathrm{Tr}^{[N\gamma]}[\exp(-2t\mathcal{L}^{X_0,F_0})]$.

One can consider generally $\ell \geqslant 2$ copies of G_0 with cyclic permutation σ, the above computations are still applicable with suitable change. Using the formula (4.32) for wave operators, one can also verify the identity (4.74) for a general class of integral kernel functions.

5. The hypoelliptic Laplacian on X

The purpose of this section is to recall the construction of the hypoelliptic Laplacian of Bismut [9, Chapter 2].

5.1. Clifford algebras

Let V be a real vector space of dimension m equipped with a real-valued nondegenerate symmetric bilinear form B. The Clifford algebra $c(V)$ of V with respect to B is the algebra generated by 1 and $a \in V$ and the relations,

$$ab + ba = -2B(a, b),\ a, b \in V. \tag{5.1}$$

We will denote by $\widehat{c}(V)$ the Clifford algebra of V associated with $-B$. Also they are $\mathbb{Z}_2$-graded algebras, we write

$$c(V) = c_+(V) \oplus c_-(V),\ \ \widehat{c}(V) = \widehat{c}_+(V) \oplus \widehat{c}_-(V). \tag{5.2}$$

Since B is nondegenerate, it induces an isomorphism φ between V and V^* such that if $a, b \in V$, then

$$\langle \varphi(a), b\rangle = B(a, b). \tag{5.3}$$

Let B^* be the corresponding bilinear form on V^*, which also extends to a nondegenerate symmetric bilinear form on $\Lambda^\bullet(V^*)$.

If $\alpha \in V^*$, $a \in V$, let $\alpha\wedge$ denote the exterior product of α acting on $\Lambda^\bullet(V^*)$, and let i_a denote the interior product (or the contraction) of a acting on $\Lambda^\bullet(V^*)$. If $a \in V$, let $c(a)$, $\widehat{c}(a) \in \mathrm{End}(\Lambda^\bullet(V^*))$ be given by

$$c(a) = \varphi(a) \wedge - i_a,\ \widehat{c}(a) = \varphi(a) \wedge + i_a. \tag{5.4}$$

Then $c(a)$, $\widehat{c}(a)$ are odd operators, which are respectively antisymmetric, symmetric with respect to B^*. If $a, b \in V$, then

$$[c(a), c(b)] = -2B(a, b),\ [\widehat{c}(a), \widehat{c}(b)] = 2B(a, b),\ [c(a), \widehat{c}(b)] = 0. \tag{5.5}$$

By (5.5), $\Lambda^\bullet(V^*)$ is a $c(V)\widehat{\otimes}\widehat{c}(V)$-module. If $D \in c(V)$ or $\widehat{c}(V)$, then we denote by $c(D)$ or $\widehat{c}(D)$ the corresponding actions on $\Lambda^\bullet(V^*)$ defined by (5.4).

Let $e_1, \ldots, e_m$ be a basis of V, and let $e_1^*, \ldots, e_m^*$ be the dual basis of V with respect to B, so that $B(e_i, e_j^*) = \delta_{ij}$. Let $e^1, \ldots, e^m$ be the basis of V^* which is dual to the basis $e_1, \ldots, e_m$. Then $e^i = \varphi(e_i^*)$.

Note that $1 \in \mathbb{R} = \Lambda^0(V^*)$. The symbol map $\boldsymbol{\sigma} : D \in \widehat{c}(V) \mapsto \widehat{c}(D) \cdot 1 \in \Lambda^\bullet(V^*)$ is an isomorphism of $\mathbb{Z}_2$-graded vector spaces. If $\alpha \in \Lambda^p(V^*)$, then the inverse map of $\boldsymbol{\sigma}$ is given by

$$\widehat{c}(\alpha) = \frac{1}{p!} \sum_{1\leqslant i_1,\ldots,i_p\leqslant m} \alpha(e_{i_1}^*, \ldots, e_{i_p}^*)\widehat{c}(e_{i_1}) \cdots \widehat{c}(e_{i_p}) \in \widehat{c}(V). \tag{5.6}$$

If $A \in \mathrm{End}(V)$ is antisymmetric with respect to B, set

$$\widehat{c}(A) = -\frac{1}{4}\sum_{i,j} B(e_i^*, e_j^*)\widehat{c}(e_i)\widehat{c}(e_j). \tag{5.7}$$

Definition 5.1. — *The number operator $N^{\Lambda^\bullet(V^*)}$ on $\Lambda^\bullet(V^*)$ is such that, if $\alpha \in \Lambda^p(V^*)$, then*

$$N^{\Lambda^\bullet(V^*)}\alpha = p\alpha. \tag{5.8}$$

One verifies easily that

$$N^{\Lambda^\bullet(V^*)} = \frac{1}{2}\sum_{i=1}^{m} c(e_i^*)\widehat{c}(e_i) + \frac{m}{2}. \tag{5.9}$$

We refer to [33, Chapter I], [6, Chapter 3] for more detailed discussions on Clifford algebras.

5.2. Harmonic oscillators

Now we consider the Lie algebra $\mathfrak{g}$ of G equipped with the bilinear form B introduced in Subsection 2.1. Let $c(\mathfrak{g})$, $\widehat{c}(\mathfrak{g})$ be the Clifford algebras associated with $(\mathfrak{g}, B)$, $(\mathfrak{g}, -B)$. By restricting B to $\mathfrak{p}$, $\mathfrak{k}$, we get the Clifford algebras $c(\mathfrak{p})$, $\widehat{c}(\mathfrak{p})$, $c(\mathfrak{k})$, $\widehat{c}(\mathfrak{k})$. By (2.1),

$$c(\mathfrak{g}) = c(\mathfrak{p})\widehat{\otimes}c(\mathfrak{k}), \qquad \widehat{c}(\mathfrak{g}) = \widehat{c}(\mathfrak{p})\widehat{\otimes}\widehat{c}(\mathfrak{k}). \tag{5.10}$$

If $a \in \mathfrak{g}$, let ∇_a denote the corresponding differentiation operator along $\mathfrak{g}$. Let $e_1, \ldots, e_m$ be an orthonormal basis of $\mathfrak{p}$, and let $e_{m+1}, \ldots, e_{m+n}$ be an orthonormal basis of $\mathfrak{k}$. If $Y \in \mathfrak{g}$, we split Y in the form $Y = Y^{\mathfrak{p}} + Y^{\mathfrak{k}}$ with $Y^{\mathfrak{p}} \in \mathfrak{p}$, $Y^{\mathfrak{k}} \in \mathfrak{k}$. Set

$$\begin{aligned} \mathcal{D}^{\mathfrak{p}} &= \sum_{j=1}^{m} c(e_j)\nabla_{e_j}, & \mathcal{E}^{\mathfrak{p}} &= \widehat{c}(Y^{\mathfrak{p}}), \\ \mathcal{D}^{\mathfrak{k}} &= -\sum_{j=m+1}^{m+n} c(e_j)\nabla_{e_j}, & \mathcal{E}^{\mathfrak{k}} &= \widehat{c}(Y^{\mathfrak{k}}). \end{aligned} \tag{5.11}$$

Since K preserves the scalar products on $\mathfrak{p}$ and $\mathfrak{k}$, the above constructions are K-equivariant. The operators $\mathcal{D}^{\mathfrak{p}}$, $\mathcal{E}^{\mathfrak{p}}$, $\mathcal{D}^{\mathfrak{k}}$, $\mathcal{E}^{\mathfrak{k}}$ are linear differential operators acting on $\Lambda^\bullet(\mathfrak{g}^*) \otimes C^\infty(\mathfrak{g})$. Moreover,

$$[\mathcal{D}^{\mathfrak{p}} + \mathcal{E}^{\mathfrak{p}}, -\mathrm{i}\mathcal{D}^{\mathfrak{k}} + \mathrm{i}\mathcal{E}^{\mathfrak{k}}] = 0. \tag{5.12}$$

Let $\Delta^{\mathfrak{g}}$ be the Euclidean Laplacian of $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$. Then by (5.5), (5.9), we get

$$\frac{1}{2}\big(\mathcal{D}^{\mathfrak{p}}+\mathcal{E}^{\mathfrak{p}}-\mathrm{i}\mathcal{D}^{\mathfrak{k}}+\mathrm{i}\mathcal{E}^{\mathfrak{k}}\big)^2 = \frac{1}{2}\big(-\Delta^{\mathfrak{g}}+|Y|^2-(m+n)\big) + N^{\Lambda^\bullet(\mathfrak{g}^*)}. \tag{5.13}$$

The kernel of the unbounded operator in (5.13) is one-dimensional line spanned by the function $\exp(-|Y|^2/2)/\pi^{(m+n)/4}$.

5.3. The Dirac operator of Kostant

Recall that $C^{\mathfrak{g}} \in U\mathfrak{g}$ is defined in (4.8) and that $\kappa^{\mathfrak{g}} \in \Lambda^3(\mathfrak{g}^*)$ is defined in (4.14). Let $\kappa^{\mathfrak{k}} \in \Lambda^3(\mathfrak{k}^*)$ be the element defined by the same formula as in (4.14) with respect to $(\mathfrak{k}, B|_{\mathfrak{k}})$. Then by (4.15), we get

$$B^*(\kappa^{\mathfrak{k}},\kappa^{\mathfrak{k}}) = \frac{1}{6}\operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]. \tag{5.14}$$

The Clifford elements $c(\kappa^{\mathfrak{g}})$, $\widehat{c}(-\kappa^{\mathfrak{g}})$, $c(\kappa^{\mathfrak{k}})$, $\widehat{c}(-\kappa^{\mathfrak{k}})$ are defined as in (5.6). If $e \in \mathfrak{k}$, let $\mathrm{ad}(e)|_{\mathfrak{p}}$ be the restriction of $\mathrm{ad}(e)$ to $\mathfrak{p}$. Then $\widehat{c}(\mathrm{ad}(e)|_{\mathfrak{p}}) \in \widehat{c}(\mathfrak{p})$. By [9, (2.7.4)], we have

$$\widehat{c}(-\kappa^{\mathfrak{g}}) = -2\sum_{i=m+1}^{m+n}\widehat{c}(e_i)\widehat{c}(\mathrm{ad}(e_i)|_{\mathfrak{p}}) + \widehat{c}(-\kappa^{\mathfrak{k}}). \tag{5.15}$$

Definition 5.2. — *Let $\widehat{D}^{\mathfrak{g}} \in \widehat{c}(\mathfrak{g}) \otimes U\mathfrak{g}$ be the Dirac operator,*

$$\widehat{D}^{\mathfrak{g}} = \sum_{i=1}^{m+n}\widehat{c}(e_i^*)e_i + \frac{1}{2}\widehat{c}(-\kappa^{\mathfrak{g}}). \tag{5.16}$$

The operator $\widehat{D}^{\mathfrak{g}}$ is called the Dirac operators of Kostant [31].

By [31] (cf. [9, Theorem 2.7.2]), we have

$$\widehat{D}^{\mathfrak{g},2} = -C^{\mathfrak{g}} - \frac{1}{4}B^*(\kappa^{\mathfrak{g}},\kappa^{\mathfrak{g}}). \tag{5.17}$$

5.4. The operator $\mathfrak{D}_b^X$

As we saw in Subsection 5.3, $\widehat{D}^{\mathfrak{g}}$ acts on $C^\infty(G,\Lambda^\bullet(\mathfrak{g}^*))$. Recall that $\mathcal{D}^{\mathfrak{p}}+\mathcal{E}^{\mathfrak{p}}-\mathrm{i}\mathcal{D}^{\mathfrak{k}}+i\mathcal{E}^{\mathfrak{k}}$ is a differential operator acting on $C^\infty(\mathfrak{g},\Lambda^\bullet(\mathfrak{g}^*))$.

Definition 5.3. — *For $b>0$, let $\mathfrak{D}_b$ be the differential operator,*

$$\mathfrak{D}_b = \widehat{D}^{\mathfrak{g}} + \mathrm{i}c([Y^{\mathfrak{k}},Y^{\mathfrak{p}}]) + \frac{1}{b}(\mathcal{D}^{\mathfrak{p}}+\mathcal{E}^{\mathfrak{p}}-\mathrm{i}\mathcal{D}^{\mathfrak{k}}+\mathrm{i}\mathcal{E}^{\mathfrak{k}}). \tag{5.18}$$

Then $\mathfrak{D}_b$ acts on $C^\infty(G\times\mathfrak{g},\Lambda^\bullet(\mathfrak{g}^*))$.

If $Y \in \mathfrak{g}$, let $\underline{Y^{\mathfrak{p}}}$, $\underline{Y^{\mathfrak{k}}}$ denote the tangent vector fields on G associated with $Y^{\mathfrak{p}}$, $Y^{\mathfrak{k}} \in \mathfrak{g}$. Let $\nabla^{\mathfrak{g}}_{[Y^{\mathfrak{k}},Y^{\mathfrak{p}}]}$ denote the differentiation operator in the direction $[Y^{\mathfrak{k}}, Y^{\mathfrak{p}}] \in \mathfrak{p}$ along the vector space $\mathfrak{g}$. The following identity is obtained in [9, Section 2.11].

Theorem 5.4. — *We have the following formula for $\mathfrak{D}_b^2$,*

$$\begin{aligned}\frac{\mathfrak{D}_b^2}{2} = \frac{\widehat{D}^{\mathfrak{g},2}}{2} + \frac{1}{2}\left|[Y^{\mathfrak{k}}, Y^{\mathfrak{p}}]\right|^2 \\ + \frac{1}{2b^2}\left(-\Delta^{\mathfrak{p}\oplus\mathfrak{k}} + |Y|^2 - m - n\right) + \frac{N^{\Lambda^\bullet(\mathfrak{g}^*)}}{b^2} \\ + \frac{1}{b}\Big(\underline{Y^{\mathfrak{p}}} + \mathrm{i}\underline{Y^{\mathfrak{k}}} - \mathrm{i}\nabla^{\mathfrak{g}}_{[Y^{\mathfrak{k}},Y^{\mathfrak{p}}]} + \widehat{c}(\mathrm{ad}(Y^{\mathfrak{p}} + \mathrm{i}Y^{\mathfrak{k}})) \\ + 2\mathrm{i}c(\mathrm{ad}(Y^{\mathfrak{k}})|_{\mathfrak{p}}) - c(\mathrm{ad}(Y^{\mathfrak{p}}))\Big).\end{aligned} \tag{5.19}$$

Recall that (E, ρ^E) is a unitary representation of K^σ. Let $C_K^\infty(G \times \mathfrak{g}, \Lambda^\bullet(\mathfrak{g}^*)\otimes E)$ denote the set of K-invariant sections. Recall that $\widehat{\pi} : \widehat{\mathcal{X}} \to X$ is the total space of $TX \oplus N$. Then we have

$$C_K^\infty\big(G \times \mathfrak{g}, \Lambda^\bullet(\mathfrak{g}^*) \otimes E\big) = C^\infty\big(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F)\big). \tag{5.20}$$

Let $Y = Y^{TX} + Y^N$, $Y^{TX} \in TX$, $Y^N \in N$ be the tautological section of $\widehat{\pi}^*(TX \oplus N)$ over $\widehat{\mathcal{X}}$.

Definition 5.5. — *Let $\mathcal{H}$ be the vector space of smooth sections over X of the vector bundle $C^\infty\big(TX \oplus N, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F)\big)$.*

We can identify $\mathcal{H}$ with $C^\infty\big(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F)\big)$. Let $\nabla^{\mathcal{H}}$ be the connection on $\mathcal{H}$ induced by the connection form $\omega^{\mathfrak{k}}$ on X.

Let $e \in \mathfrak{k}$, then the vector field $[e, Y]$ on $\mathfrak{g}$ is a Killing vector field. Let $L^V_{[e,Y]}$ be the Lie derivative acting on $C^\infty(\mathfrak{g}, \Lambda^\bullet(\mathfrak{g}^*))$. Then by [9, (2.12.4)],

$$L^V_{[e,Y]} = \nabla_{[e,Y]} - (c + \widehat{c})(\mathrm{ad}(e)). \tag{5.21}$$

Note that $\widehat{D}^{\mathfrak{g}}$ is K-invariant. Let $\widehat{D}^{\mathfrak{g},X}$ be the corresponding differential operators on the smooth sections of $\mathcal{H}$. By [9, Theorem 2.12.2],

$$\widehat{D}^{\mathfrak{g},X} = \sum_{i=1}^{m} \widehat{c}(e_i)\nabla^{\mathcal{H}}_{e_i} - \sum_{j=m+1}^{m+n} \widehat{c}(e_j)\big(L^V_{[e_j,Y]} + \widehat{c}(\mathrm{ad}(e_j)|_{\mathfrak{p}}) - \rho^E(e_j)\big) + \frac{1}{2}\widehat{c}(-\kappa^{\mathfrak{k}}). \tag{5.22}$$

Let $\mathcal{D}^{TX}$, $\mathcal{E}^{TX}$, $\mathcal{D}^N$, $\mathcal{E}^N$ be the operators acting on $\widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*)\otimes F)$ along the fibre $\widehat{\mathcal{X}}$ induced by $\mathcal{D}^{\mathfrak{p}}$, $\mathcal{E}^{\mathfrak{p}}$, $\mathcal{D}^{\mathfrak{k}}$, $\mathcal{E}^{\mathfrak{k}}$. Then $\mathfrak{D}_b$ defined in (5.18) descends to an operator $\mathfrak{D}_b^X$ on $C^\infty(TX \oplus N, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F))$. Then

$$\mathfrak{D}_b^X = \widehat{D}^{\mathfrak{g},X} + \mathrm{i}c([Y^N, Y^{TX}]) + \frac{1}{b}(\mathcal{D}^{TX} + \mathcal{E}^{TX} - \mathrm{i}\mathcal{D}^N + \mathrm{i}\mathcal{E}^N). \tag{5.23}$$

5.5. The hypoelliptic Laplacian

Recall that $A \in \mathrm{End}(E)$ is a K^σ-invariant such that it gives a parallel section of $\mathrm{End}(F)$ on X. Recall that for $t > 0$, $p_t^X(x, x')$ is the heat kernel of $\mathcal{L}_A^X$.

Let $(\cdot\,,\cdot)$ denote the Hermitian metric on $\Lambda^\bullet(T^*X \oplus N^*) \otimes F$ associated with $\langle\cdot\,,\cdot\rangle$ and g^F. The Cartan involution θ acts on $\widehat{\mathcal{X}}$, so that

$$\theta(Y^{TX} + Y^N) = -Y^{TX} + Y^N. \tag{5.24}$$

Let $\mathrm{d}v_{\widehat{\mathcal{X}}}$ be the volume form on $\widehat{\mathcal{X}}$ coming from the Riemann metric on X and the Euclidean scalar product on $TX \oplus N$. Let $\eta(\cdot\,,\cdot)$ be the Hermitian form on the space of smooth compactly supported sections of $\widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F)$ over $\widehat{\mathcal{X}}$,

$$\eta(s, s') = \int_{\widehat{\mathcal{X}}} (s \circ \theta, s')\mathrm{d}v_{\widehat{\mathcal{X}}}. \tag{5.25}$$

As in [9, Sections 2.12 and 2.13], for $b > 0$, we put

$$\mathcal{L}_b^X = -\frac{1}{2}\widehat{D}^{\mathfrak{g},X,2} + \frac{1}{2}\mathfrak{D}_b^{X,2}. \tag{5.26}$$

It acts on $C^\infty\big(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F)\big)$, whose formula is given as follows,

$$\begin{aligned} \mathcal{L}_b^X = {} & \frac{1}{2}\left|[Y^N, Y^{TX}]\right|^2 \\ & + \frac{1}{2b^2}\left(-\Delta^{TX\oplus N} + |Y|^2 - m - n\right) + \frac{N^{\Lambda^\bullet(T^*X\oplus N^*)}}{b^2} \\ & + \frac{1}{b}\Big(\nabla^{\mathcal{H}}_{Y^{TX}} + \widehat{c}\big(\mathrm{ad}(Y^{TX})\big) - c\big(\mathrm{ad}(Y^{TX}) + \mathrm{i}\theta\,\mathrm{ad}(Y^N)\big) - \mathrm{i}\rho^E(Y^N)\Big). \end{aligned} \tag{5.27}$$

By Hörmander's theorem [24], both $\mathcal{L}_b^X$ and $\frac{\partial}{\partial t} + \mathcal{L}_b^X$ are hypoelliptic. The operator $\mathcal{L}_b^X$ is called the hypoelliptic Laplacian associated with (G, K). Moreover, it is formally self-adjoint with respect to $\eta(\cdot\,,\cdot)$.

By [9, Proposition 2.15.1], we have the identity

$$[\mathfrak{D}_b^X, \mathcal{L}_b^X] = 0. \tag{5.28}$$

Since σ preserves B and the splitting (2.1), both $\widehat{D}^{\mathfrak{g},X}$ and $\mathfrak{D}_b^X$ commute with G^σ, so that $\mathcal{L}_b^X$ commutes with G^σ. The section A lifts to $\widehat{\mathcal{X}}$. Let $\mathcal{L}_{A,b}^X$ be the operator acting on $C^\infty(\widehat{\mathcal{X}}, \hat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F))$ given by

$$\mathcal{L}_{A,b}^X = \mathcal{L}_b^X + A. \tag{5.29}$$

In [9, Sections 4.5 and 11.8], the heat operator $\exp(-t\mathcal{L}_{A,b}^X))$ is well-defined for $b > 0, t > 0$ with a smooth kernel $q_{b,t}^X((x,Y),(x',Y'))$.

Let $\mathbf{P}$ be the projection from $\Lambda^\bullet(T^*X \oplus e^*) \otimes F$ on $\Lambda^0(T^*X \oplus e^*) \otimes F$. For $t > 0$ and $(x,Y),(x',Y') \in \widehat{\mathcal{X}}$, put

$$\begin{aligned} q_{0,t}^X&\big((x,Y),(x',Y')\big) \\ &= \mathbf{P} p_t^X(x,x')\pi^{-(m+n)/2} \exp\left(-\frac{1}{2}\big(|Y|^2 + |Y'|^2\big)\right)\mathbf{P}. \end{aligned} \tag{5.30}$$

We recall a result established in [9, Theorem 4.5.2 and Chapter 14].

Theorem 5.6. — *Given $M \geqslant \epsilon > 0$, there exist $C, C' > 0$ such that for $0 < b \leqslant M, \epsilon \leqslant t \leqslant M, (x,Y),(x',Y') \in \widehat{\mathcal{X}}$,*

$$\left|q_{b,t}^X\big((x,Y),(x',Y')\big)\right| \leqslant C \exp\Big(-C'\big(d^2(x,x') + |Y|^2 + |Y'|^2\big)\Big). \tag{5.31}$$

As $b \to 0$, we have the uniform convergence on compact subsets of $\widehat{\mathcal{X}} \times \widehat{\mathcal{X}}$,

$$q_{b,t}^X\big((x,Y),(x',Y')\big) \to q_{0,t}^X\big((x,Y),(x',Y')\big). \tag{5.32}$$

Example 5.7 (A simple example of the hypoelliptic Laplacian). — A simple example of our setting is the real line $\mathbb{R}$ with additive Lie group structure. In this case, $G = \mathbb{R}$, $K = 0$, so that $X = \mathbb{R}$ with the standard Euclidean metric. Let $x \in \mathbb{R}$ denote the global coordinate of X, and let $y = y\frac{\partial}{\partial x}$ denote the coordinate along the tangent vector space of X. Then $\widehat{\mathcal{X}} = \mathcal{X} = \mathbb{R}_x \times \mathbb{R}_y$ is just the total space of $T\mathbb{R}$, where the subscripts x, y indicate the respective coordinates. By (5.27), the operator $\mathcal{L}_b^{\mathbb{R}}$ acting on $C^\infty(\mathbb{R}_x \times \mathbb{R}_y, \Lambda^\bullet(\mathbb{R}_y^*))$ is given by

$$\mathcal{L}_b^{\mathbb{R}} = \frac{1}{2b^2}(-\Delta_y + y^2 - 1) + \frac{N^{\Lambda^\bullet(\mathbb{R}_y^*)}}{b^2} + \frac{1}{b} y \frac{\partial}{\partial x}. \tag{5.33}$$

Note that $\frac{\partial}{\partial t} + \mathcal{L}_b^{\mathbb{R}}$ is just the Kolmogorov operator ([30], up to a conjugation). The heat kernel of $\mathcal{L}_b^{\mathbb{R}}$ has an explicit expression given in [9, Subsection 10.5], so that the convergence (5.32) can be verified directly. Here, we would like to give another straightforward computation to understand this convergence.

The geodesic flow φ_t, $t \in \mathbb{R}$ on $\mathcal{X}$ is given by $\varphi_t(x,y) = (x + ty, y)$. For $f \in C^\infty(\mathbb{R}_x, \mathbb{R})$, we identify it with the section $f(x)\frac{1}{\pi^{1/4}}\exp(-y^2/2) \in$

$C^\infty(\mathbb{R}_x \times \mathbb{R}_y, \Lambda^\bullet(\mathbb{R}_y^*))$. This identification preserves the L_2-metrics for L_2-functions. A direct computation shows that for $b > 0$,

$$\mathcal{L}_b^{\mathbb{R}} \varphi_{-b}^* \left(f(x) \frac{1}{\pi^{1/4}} \,\mathrm{e}^{-y^2/2} \right) = \varphi_{-b}^* \left(-\frac{1}{2} \Delta_x(f) \frac{1}{\pi^{1/4}} \,\mathrm{e}^{-y^2/2} \right). \tag{5.34}$$

This gives an explicit relation, conjugation by the geodesic flow, between the hypoelliptic Laplacian $\mathcal{L}_b^{\mathbb{R}}$ and the elliptic Laplacian $-\frac{1}{2}\Delta_x$ on $X = \mathbb{R}$. If we take $b \to 0$ in (5.34), it explains well the convergence in (5.32).

6. A proof of Theorem 4.6

The purpose of this section is to establish Theorem 4.6. The geometric constructions in Sections 2 and 3 play important roles in the proof. In particular, due to the geometric formulations of the twisted orbital supertrace $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_{A,b}^X)]$, the local index techniques used in [9, Chapter 9] are still applicable to compute explicitly its limit as $b \to \infty$. Therefore, our proof to Theorem 4.6 is partly derived from [9, Chapter 9].

6.1. A fundamental identity for twisted orbital supertraces

Recall that $\mathcal{L}_A^X$, $\mathcal{L}_{A,b}^X$ were defined in Subsections 4.2 and 5.5, and that $p_t^X, q_{b,t}^X$ are the associated elliptic and hypoelliptic heat kernels. Using (5.31) and the fact that $\mathcal{L}_{A,b}^X$ commutes with σ, for $b > 0$, $t > 0$, $q_{b,t}^X \in \mathfrak{Q}^{\sigma,\infty}$ (cf. Definition 3.6). By the results of Subsection 3.3, $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_{A,b}^X)]$ is well-defined. The following theorem extends [9, Theorem 4.6.1].

Theorem 6.1. — *For any $b > 0, t > 0$, the following identity holds,*

$$\mathrm{Tr_s}^{[\gamma\sigma]}\left[\exp(-t\mathcal{L}_{A,b}^X)\right] = \mathrm{Tr}^{[\gamma\sigma]}\left[\exp(-t\mathcal{L}_A^X)\right]. \tag{6.1}$$

Proof. — By (3.38) and using Proposition 3.9, we get

$$\frac{\partial}{\partial b} \mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_{A,b}^X)] = -t\,\mathrm{Tr_s}^{[\gamma\sigma]}\left[\left(\frac{\partial}{\partial b}\mathcal{L}_{A,b}^X\right)\exp(-t\mathcal{L}_{A,b}^X)\right]. \tag{6.2}$$

By (5.26) and (5.28), we have

$$\frac{\partial}{\partial b}\mathcal{L}_{A,b}^X = \frac{1}{2}\left[\mathfrak{D}_b^X, \frac{\partial}{\partial b}\mathfrak{D}_b^X\right], \quad [\mathfrak{D}_b^X, \mathcal{L}_{A,b}^X] = 0. \tag{6.3}$$

Then we get

$$\frac{\partial}{\partial b}\operatorname{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_{A,b})] = -\frac{t}{2}\operatorname{Tr_s}^{[\gamma\sigma]}\left[\left[\mathfrak{D}^X_b, \frac{\partial}{\partial b}\mathfrak{D}^X_b\right]\exp(-t\mathcal{L}^X_{A,b})\right] = -\frac{t}{2}\operatorname{Tr_s}^{[\gamma\sigma]}\left[\left[\mathfrak{D}^X_b, \left(\frac{\partial}{\partial b}\mathfrak{D}^X_b\right)\exp(-t\mathcal{L}^X_{A,b})\right]\right]. \tag{6.4}$$

Applying again Proposition 3.9, we get

$$\frac{\partial}{\partial b}\operatorname{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_{A,b})] = 0. \tag{6.5}$$

Now we only need to prove that

$$\lim_{b\to 0}\operatorname{Tr_s}^{[\gamma\sigma]}\left[\exp(-t\mathcal{L}^X_{A,b})\right] = \operatorname{Tr}^{[\gamma\sigma]}\left[\exp(-t\mathcal{L}^X_{A})\right]. \tag{6.6}$$

By (2.73) and Theorem 5.6, given $t > 0$, there exist $C, C' > 0$ such that for $0 < b \leqslant 1, f \in \mathfrak{p}^{\perp}_{\sigma}(\gamma), Y \in (TX \oplus N)_{\mathrm{e}^f p1}$,

$$\left|q^X_{b,t}\big((\mathrm{e}^f p1, Y), \gamma\sigma(\mathrm{e}^f p1, Y)\big)\right| \leqslant C\exp\big(-C'(|f|^2 + |Y|^2)\big) \tag{6.7}$$

Using (3.20), (3.38), (5.32) and dominated convergence, we get (6.6). The proof of our theorem is completed. □

6.2. An identity for $J_{\gamma\sigma}(Y^{\mathfrak{k}}_0)$

Recall that $p = \dim \mathfrak{p}_\sigma(\gamma)$, $q = \dim \mathfrak{k}_\sigma(\gamma)$, $r = \dim \mathfrak{z}_\sigma(\gamma) = p + q$. Let $e_1, \dots, e_p$ be an orthonormal basis of $\mathfrak{p}_\sigma(\gamma)$, and let $e_{p+1}, \dots, e_r$ be an orthonormal basis of $\mathfrak{k}_\sigma(\gamma)$. Let $e^1, \dots, e^r$ be the corresponding dual basis of $\mathfrak{z}_\sigma(\gamma)^*$. Let $\underline{\mathfrak{z}_\sigma}(\gamma)$, $\underline{\mathfrak{z}_\sigma}(\gamma)^*$ be another copies of $\mathfrak{z}_\sigma(\gamma)$, $\mathfrak{z}_\sigma(\gamma)^*$. We underline the obvious objects associated with $\underline{\mathfrak{z}_\sigma}(\gamma)$, $\underline{\mathfrak{z}_\sigma}(\gamma)^*$. Let $c(\mathfrak{z}_\sigma(\gamma))$ denote the Clifford algebra associated with $(\mathfrak{z}_\sigma(\gamma), B|_{\mathfrak{z}_\sigma(\gamma)})$, we also identify the elements in $c(\mathfrak{z}_\sigma(\gamma))$ with their actions on $\Lambda^\bullet(\mathfrak{g}^*)$ given in (5.4).

By (2.1), we get

$$\mathfrak{p} \times \mathfrak{g} = \mathfrak{p} \times (\mathfrak{p} \oplus \mathfrak{k}). \tag{6.8}$$

We denote by y the tautological section of the first copy of $\mathfrak{p}$ in the right-hand side of (6.8), and by $Y^{\mathfrak{g}} = Y^{\mathfrak{p}} + Y^{\mathfrak{k}}$ the tautological section of $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We also denote by $\mathrm{d}y$, $\mathrm{d}Y^{\mathfrak{g}} = \mathrm{d}Y^{\mathfrak{p}}\mathrm{d}Y^{\mathfrak{k}}$ the volume forms on $\mathfrak{p}$, $\mathfrak{g}$ respectively. Recall that $\Delta^{\mathfrak{p}\oplus\mathfrak{k}}$ is the standard Laplacian on $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, i.e., the second factor in the right-hand side of (6.8). Let ∇^H denote differentiation in the variable $y \in \mathfrak{p}$, and let ∇^V denote the differentiation in the variable $Y^{\mathfrak{g}} \in \mathfrak{g}$.

Put

$$\alpha = \sum_{i=1}^{r} c(e_i)\underline{e}^i \in c(\mathfrak{z}_\sigma(\gamma))\widehat{\otimes}\Lambda^\bullet(\underline{\mathfrak{z}_\sigma}(\gamma)^*). \tag{6.9}$$

As an analogue in [9, Section 5.1], if $Y_0^{\mathfrak{k}} \in \mathfrak{k}_\sigma(\gamma)$, let $\mathcal{P}_{a,Y_0^{\mathfrak{k}}}$ be the differential operator acting on $C^\infty(\mathfrak{p}\times\mathfrak{g}, \Lambda^\bullet(\mathfrak{g}^*)\widehat{\otimes}\Lambda^\bullet(\underline{\mathfrak{z}_\sigma}(\gamma)^*))$ defined as follows,

$$\begin{aligned}\mathcal{P}_{a,Y_0^{\mathfrak{k}}} = \frac{1}{2}\left|[Y^{\mathfrak{k}},a]+[Y_0^{\mathfrak{k}},Y^{\mathfrak{p}}]\right|^2 - \frac{1}{2}\Delta^{\mathfrak{p}\oplus\mathfrak{k}} + \alpha - \nabla^H_{Y^{\mathfrak{p}}} \\ - \nabla^V_{[a+Y_0^{\mathfrak{k}},[a,y]]} - \widehat{c}(\mathrm{ad}(a)) + c\left(\mathrm{ad}(a) + \mathrm{i}\theta\,\mathrm{ad}(Y_0^{\mathfrak{k}})\right).\end{aligned} \tag{6.10}$$

By Hörmander's theorem [24], both $\mathcal{P}_{a,Y_0^{\mathfrak{k}}}$, $\frac{\partial}{\partial t} + \mathcal{P}_{a,Y_0^{\mathfrak{k}}}$ are hypoelliptic.

Let $R_{Y_0^{\mathfrak{k}}}$ be the smooth kernel of $\exp(-\mathcal{P}_{a,Y_0^{\mathfrak{k}}})$ with respect to the volume $\mathrm{d}y\,\mathrm{d}Y^{\mathfrak{g}}$ on $\mathfrak{p}\times\mathfrak{g}$. Then by [9, (5.1.10)], for $(y,Y^{\mathfrak{g}}),(y',Y^{\mathfrak{g}\prime}) \in \mathfrak{p}\times\mathfrak{g}$,

$$R_{Y_0^{\mathfrak{k}}}\left((y,Y^{\mathfrak{g}}),(y',Y^{\mathfrak{g}\prime})\right) \in \mathrm{End}(\Lambda^\bullet(\mathfrak{z}_\sigma^\perp(\gamma)^*))\widehat{\otimes}c(\mathfrak{z}_\sigma(\gamma))\widehat{\otimes}\Lambda^\bullet(\underline{\mathfrak{z}_\sigma}(\gamma)^*). \tag{6.11}$$

Definition 6.2. — *Let $\widehat{\mathrm{Tr}_{\mathrm{s}}}$ denote the supertrace (linear) functional on $c(\mathfrak{z}_\sigma(\gamma))\widehat{\otimes}\Lambda^\bullet(\underline{\mathfrak{z}_\sigma}(\gamma)^*)$ such that it vanishes on monomials of nonmaximal length, and gives the value $(-1)^r$ to the monomial $c(e_1)\underline{e}^1\cdots c(e_r)\underline{e}^r$. It also extends to a supertrace functional on*

$$\mathrm{End}(\Lambda^\bullet(\mathfrak{z}_\sigma^\perp(\gamma)^*))\widehat{\otimes}c(\mathfrak{z}_\sigma(\gamma))\widehat{\otimes}\Lambda^\bullet(\underline{\mathfrak{z}_\sigma}(\gamma)^*)$$

by tensoring with the supertrace on $\mathrm{End}(\Lambda^\bullet(\mathfrak{z}_\sigma^\perp(\gamma)^*))$. We still denote it by $\widehat{\mathrm{Tr}_{\mathrm{s}}}$.

Now we give the important result established in [9, Theorem 5.5.1].

Proposition 6.3. — *For $Y_0^{\mathfrak{k}} \in \mathfrak{k}_\sigma(\gamma)$, we have*

$$\begin{aligned}J_{\gamma\sigma}(Y_0^{\mathfrak{k}}) = (2\pi)^{r/2}\int_{y\in\mathfrak{p}_\sigma^\perp(\gamma),Y^{\mathfrak{g}}\in\mathfrak{p}\oplus\mathfrak{k}_\sigma^\perp(\gamma)} \widehat{\mathrm{Tr}_{\mathrm{s}}}\Big[\mathrm{Ad}(k^{-1}\sigma) \\ R_{Y_0^{\mathfrak{k}}}\Big((y,Y^{\mathfrak{g}}),\mathrm{Ad}(k^{-1}\sigma)(y,Y^{\mathfrak{g}})\Big)\Big]\mathrm{d}y\,\mathrm{d}Y^{\mathfrak{g}}.\end{aligned} \tag{6.12}$$

Proof. — In the proof of [9, Theorem 5.5.1], the computations of the supertrace functional in the right-hand side of (6.12) only depend on the adjoint actions of γ, k^{-1} and a and the fact that they commute with each other. Therefore, when replacing γ, k^{-1} by $\gamma\sigma$, $k^{-1}\sigma$, the computations in [9, Chapter 5] still hold, so that (6.12) holds. ☐

6.3. A proof of Theorem 4.6

Recall that $\widehat{\mathcal{X}}$ is the total space of the vector bundle $TX \oplus N \to X$, which can be canonically identified with $X \times \mathfrak{g}$ as in (2.9). For $b \neq 0$, $s(x,Y) \in C^\infty\big(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\bullet(T^*X \oplus N^*) \otimes F)\big)$, set

$$F_b s(x,Y) = s(x,bY). \tag{6.13}$$

For $t > 0$, we denote with an extra subscript t the hypoelliptic Laplacian defined in Subsection 5.5 associated with the bilinear form B/t. By (5.27), we have

$$F_{\sqrt{t}} t^{N^{\Lambda^\bullet(T^*X\oplus N^*)/2}} \mathcal{L}^X_{b,t} t^{-N^{\Lambda^\bullet(T^*X\oplus N^*)/2}} F^{-1}_{\sqrt{t}} = t\mathcal{L}^X_{\sqrt{t}b}. \tag{6.14}$$

By Remark (4.2) and (6.14), it is enough to prove (4.18) with $t = 1$. When $t = 1$, we will drop the subscript t in the corresponding notation of heat kernels, such as $q^X_b = q^X_{b,1}$.

By (6.1), we will make $b \to +\infty$ in $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-\mathcal{L}^X_{A,b})]$. Generally speaking, all the analytic and geometric constructions of [9] only depend on the fact that G acts on X as a group of isometries, replacing G by G^σ does not change anything from that point of view. This is why we will freely use the arguments in [9, Chapter 9]. We sketch the main steps of the proof as follows.

At first, we introduce the $\gamma\sigma$-periodic points of the geodesic flow on $\widehat{\mathcal{X}}$. Let $\{\varphi_t\}_{t\in\mathbb{R}}$ denote the geodesic flow on $\mathcal{X}$ associated with g^{TX}. The flow $\{\varphi_t\}_{t\in\mathbb{R}}$ lifts to a flow of diffeomorphisms of $\widehat{\mathcal{X}}$. If $(x, Y^{TX}, Y^N) \in \widehat{\mathcal{X}}$, set

$$(x_t, Y^{TX}_t, Y^N_t) = \varphi_t(x, Y^{TX}, Y^N). \tag{6.15}$$

Then x_t is just the geodesic starting at x with speed Y^{TX}_t, and Y^N_t is the parallel transport of Y^N along x_t. Set

$$\widehat{\mathcal{F}}_{\gamma\sigma} = \{z \in \widehat{\mathcal{X}} \;:\; (\gamma\sigma)^{-1}\varphi_1 z = z\}. \tag{6.16}$$

The vector $a \in \mathfrak{p}$ defines a constant section of $X \times \mathfrak{g}$. By (2.9), we can view a as a smooth section of $TX \oplus N$. Let a^{TX}, a^N the corresponding parts of this section in TX, N respectively. In the global coordinate system $(\mathfrak{p}, \exp_{x_0})$ of X defined in Subsection 2.1, for $Y^\mathfrak{p} \in \mathfrak{p}$, by [9, Proposition 3.2.4], we have

$$a^{TX}(Y^\mathfrak{p}) = \cosh(\mathrm{ad}(Y^\mathfrak{p}))a. \tag{6.17}$$

Set

$$\begin{aligned} N_\sigma(k^{-1}) &= Z^0_\sigma(\gamma) \times_{K^0_\sigma(\gamma)} \mathfrak{k}_\sigma(k^{-1}) \\ &= \{Y^N \in N|_{X(\gamma\sigma)} \;:\; \mathrm{Ad}(k^{-1}\sigma)Y^N = Y^N\} \end{aligned} \tag{6.18}$$

Then we have

$$\widehat{\mathcal{F}}_{\gamma\sigma}=\left\{\left(x,a^{TX}(x),Y^N\right)\in\widehat{\mathcal{X}}\ :\ (x,Y^N)\in N_\sigma(k^{-1}),x\in X(\gamma\sigma)\right\}. \tag{6.19}$$

Put

$$\underline{\mathcal{L}}^X_{A,b}=F_{-b}\mathcal{L}^X_{A,b}F^{-1}_{-b}. \tag{6.20}$$

Let $\underline{q}^X_b$ be the kernel associated with $\exp(-\underline{\mathcal{L}}^X_{A,b})$ with respect to $\mathrm{d}x\mathrm{d}Y$. By (3.38), we can use $\underline{q}^X_b$ instead of q^X_b to define $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-\mathcal{L}^X_{A,b})]$.

An important property of the hypoelliptic heat kernel proved in [9, Theorem 9.1.1], after adapting to our twisted case, is that for the points $(x,Y)\in\widehat{\mathcal{X}}$ away from $\widehat{\mathcal{F}}_{\gamma\sigma}$, the norm of $\underline{q}^X_b((x,Y),\gamma\sigma(x,Y))$ decays to 0 exponentially with respect to $d_{\gamma\sigma}(x)$ and $|Y|$ (cf. (5.31)). As a consequence, by the arguments in [9, Section 9.2], if $\beta\in\,]0,1]$ is fixed, $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-\mathcal{L}^X_{A,b})]$ is given by the limit as $b\to+\infty$ of the following integral,

$$\int_{\substack{f\in\mathfrak{p}^\perp_\sigma(\gamma),\ |f|\leqslant\beta;\\ Y\in\ \mathfrak{g},\\ |Y^{TX}-a^{TX}|\leqslant\beta.}}\mathrm{Tr_s}^{\Lambda^\bullet(T^*X\oplus N^*)\otimes F}\Big[\gamma\sigma\underline{q}^X_b\left((p\,\mathrm{e}^f,Y),\right.\\ \left.\gamma\sigma(p\,\mathrm{e}^f,Y)\right)\Big]r(f)\,\mathrm{d}f\,\mathrm{d}Y. \tag{6.21}$$

Recall that $\mathrm{d}Y=\mathrm{d}Y^{TX}\mathrm{d}Y^N$. Put $N_\sigma(\gamma)=Z^0_\sigma(\gamma)\times_{K^0_\sigma(\gamma)}\mathfrak{k}_\sigma(\gamma)$ the vector bundle on $X(\gamma\sigma)$. Let $N^\perp_\sigma(\gamma)$ be the orthogonal bundle of $N_\sigma(\gamma)$ in $N|_{X(\gamma\sigma)}$, then $N^\perp_\sigma(\gamma)=Z^0_\sigma(\gamma)\times_{K^0_\sigma(\gamma)}\mathfrak{k}^\perp_\sigma(\gamma)$. Recall that the projection $P_{\gamma\sigma}:X\to X(\gamma\sigma)$ is described in Theorem 2.13.

We trivialize the vector bundles TX, N by parallel transport along the geodesics orthogonal to $X(\gamma\sigma)$ with respect to ∇^{TX},∇^N, then the vector bundles TX, N on X can be identified with $P^*_{\gamma\sigma}(TX|_{X(\gamma\sigma)})$, $P^*_{\gamma\sigma}(N|_{X(\gamma\sigma)})$. If $f\in\mathfrak{p}^\perp_\sigma(\gamma)$, at $\rho_{\gamma\sigma}(1,f)$, we may write $Y^N\in N$ in the form

$$Y^N=Y^{\mathfrak{k}}_0+Y^{N,\perp},\ Y^{\mathfrak{k}}_0\in\mathfrak{k}_\sigma(\gamma),\ Y^{N,\perp}\in\mathfrak{k}^\perp_\sigma(\gamma). \tag{6.22}$$

Let $\mathrm{d}Y^{\mathfrak{k}}_0$, $\mathrm{d}Y^{N,\perp}$ be the volume elements on $\mathfrak{k}_\sigma(\gamma)$, $\mathfrak{k}^\perp_\sigma(\gamma)$, so that $\mathrm{d}Y^N=\mathrm{d}Y^{\mathfrak{k}}_0\mathrm{d}Y^{N,\perp}$. We rewrite the integral in (6.21) as follows,

$$b^{-4m-2n+2r}\int_{\substack{f\in\mathfrak{p}^\perp_\sigma(\gamma),\ |f|\leqslant b^2\beta\\ Y\in\ \mathfrak{g},\ |Y^{TX}|\leqslant b^2\beta}}\mathrm{Tr_s}^{\Lambda^\bullet(T^*X\oplus N^*)\otimes F}\\ \left[\gamma\sigma\underline{q}^X_b\left(\left(p\,\mathrm{e}^{f/b^2},a^{TX}+\frac{Y^{TX}}{b^2},Y^{\mathfrak{k}}_0+\frac{Y^{N,\perp}}{b^2}\right),\right.\right.\\ \left.\left.\gamma\sigma\left(p\,\mathrm{e}^{f/b^2},a^{TX}+\frac{Y^{TX}}{b^2},Y^{\mathfrak{k}}_0+\frac{Y^{N,\perp}}{b^2}\right)\right)\right]r\left(\frac{f}{b^2}\right)\mathrm{d}f\mathrm{d}Y^{TX}\mathrm{d}Y^{\mathfrak{k}}_0\mathrm{d}Y^{N,\perp}. \tag{6.23}$$

Now we deal with the factor b^{2r} in (6.23) as in [9, Sections 9.3–9.5]. Recall that α is defined in (6.9). By (2.9), $TX \oplus N$ is identified with the trivial vector bundle $\mathfrak{g}$ on X. Let $(TX \oplus N)_\sigma(\gamma)$ be the subbundle of $TX \oplus N$ corresponding $\mathfrak{z}_\sigma(\gamma) \subset \mathfrak{g}$, and let $(\underline{TX \oplus N})_\sigma(\gamma)^*$ be one copy of the dual bundle of $(TX \oplus N)_\sigma(\gamma)$. We regard α as a constant section of the trivial bundle $c(\mathfrak{g}) \otimes \underline{\mathfrak{z}_\sigma}(\gamma)^*$, hence a section of $c(TX \oplus N) \otimes (\underline{TX \oplus N})_\sigma(\gamma)^*$.

Set

$$\mathfrak{L}^X_{A,b} = \underline{\mathcal{L}}^X_{A,b} + \alpha. \tag{6.24}$$

It acts on $C^\infty\big(\widehat{\mathcal{X}}, \widehat{\pi}^*\big(\Lambda^\bullet(T^*X \oplus N^*) \otimes F \widehat{\otimes} \Lambda^\bullet((\underline{TX \oplus N})_\sigma(\gamma)^*)\big)\big)$. Note that $\mathfrak{L}^X_{A,b}$ commutes with the action of $\gamma\sigma$, e^a and $k^{-1}\sigma$. Let $\mathfrak{q}^X_b$ be the smooth kernel of $\exp(-\mathfrak{L}^X_{A,b})$ with respect to the volume $\mathrm{d}x\mathrm{d}Y$.

We extend the basis $\{e_i\}_{i=1}^r$ of $\mathfrak{z}_\sigma(\gamma)$ to an orthonormal basis $\{e_i\}_{i=1}^{m+n}$ of $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$. Since $\mathrm{End}(\Lambda^\bullet(\mathfrak{g}^*)) = c(\mathfrak{g}) \widehat{\otimes} \widehat{c}(\mathfrak{g})$, we can extend $\widehat{\mathrm{Tr}_\mathrm{s}}$ in Definition 6.2 to a linear functional on $\mathrm{End}(\Lambda^\bullet(\mathfrak{g}^*)) \widehat{\otimes} \Lambda^\bullet(\underline{\mathfrak{z}_\sigma}(\gamma)^*)$ by making it vanish on all the monomials in the $c(e_i)$, $\widehat{c}(e_i)$, $\underline{e}^k$, $1 \leqslant i \leqslant m+n$, $1 \leqslant k \leqslant r$ except on

$$c(e_1)\underline{e}^1 \cdots c(e_r)\underline{e}^r c(e_{r+1})\widehat{c}(e_{r+1}) \cdots c(e_{m+n})\widehat{c}(e_{m+n}). \tag{6.25}$$

Moreover,

$$\widehat{\mathrm{Tr}_\mathrm{s}}\big[c(e_1)\underline{e}^1 \cdots c(e_r)\underline{e}^r c(e_{r+1})\widehat{c}(e_{r+1}) \cdots c(e_{m+n})\widehat{c}(e_{m+n})\big] = (-1)^{r+n-q}(-2)^{m+n-r}. \tag{6.26}$$

The map $\widehat{\mathrm{Tr}_\mathrm{s}}$ also extends to a linear functional on

$$\mathrm{End}(\Lambda^\bullet(\mathfrak{g}^*)) \widehat{\otimes} \Lambda^\bullet(\underline{\mathfrak{z}_\sigma}(\gamma)^*) \otimes \mathrm{End}(E)$$

by tensoring with $\mathrm{Tr}^E[\cdot]$.

By [9, Theorem 9.5.2 and Proposition 9.5.4], the operator $\mathfrak{L}^X_{A,b}$ is conjugate to $\underline{\mathcal{L}}^X_{A,b}$, and if $(x,Y) \in \widehat{\mathcal{X}}$, then

$$\mathrm{Tr}_\mathrm{s}\left[\gamma\sigma \underline{q}^X_b\big((x,Y), \gamma\sigma(x,Y)\big)\right] = b^{-2r}\widehat{\mathrm{Tr}_\mathrm{s}}\big[\gamma\sigma \mathfrak{q}^X_b\big((x,Y), \gamma\sigma(x,Y)\big)\big]. \tag{6.27}$$

Now we proceed as in [9, Sections 9.8–9.11], we can establish an analog of [9, Theorem 9.6.1], which says that as $b \to +\infty$,

$$\begin{aligned} b^{-4m-2n}\gamma\sigma \mathfrak{q}^X_b\Big(&\big(p\,\mathrm{e}^{f/b^2}, a^{TX} + Y^{TX}/b^2, Y^{\mathfrak{k}}_0 + Y^{N,\perp}/b^2\big), \\ &\gamma\sigma\big(p\,\mathrm{e}^{f/b^2}, a^{TX} + Y^{TX}/b^2, Y^{\mathfrak{k}}_0 + Y^{N,\perp}/b^2\big)\Big) \\ \to \exp\big(-|a|^2/2 - |Y^{\mathfrak{k}}_0|^2/2\big)\,&\mathrm{Ad}(k^{-1}\sigma)R_{Y^0_{\mathfrak{k}}}\big((f,Y), \mathrm{Ad}(k^{-1}\sigma)(f,Y)\big) \\ &\cdot \rho^E(k^{-1}\sigma)\exp\big(-\mathrm{i}\rho^E(Y^{\mathfrak{k}}_0) - A\big). \end{aligned} \tag{6.28}$$

As in [9, Theorem 9.5.6], there exist $C_\beta > 0$, $C_{\gamma\sigma,\beta} > 0$ such that for $b \geqslant 1$, $f \in \mathfrak{p}_\sigma^\perp(\gamma)$, $|f| \leqslant b^2\beta$, $|Y^{TX}| \leqslant b^2\beta$, then left-hand side of (6.28) is bounded by

$$C_\beta \exp\Big(-C_{\gamma\sigma,\beta}\big(|f|^2 + |Y^{TX}|^2 + |Y_0^{\mathfrak{k}}|^2 + |Y^{N,\perp}|\big)\Big). \tag{6.29}$$

Combining (6.23) and (6.27)–(6.29), we get

$$\lim_{b\to+\infty} \mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-\mathcal{L}_{A,b}^X)] = \exp(-|a|^2/2) \int_{\substack{(y,Y^{\mathfrak{g}},Y_0^{\mathfrak{k}}) \\ \in \mathfrak{p}_\sigma^\perp(\gamma)\times\left(\mathfrak{p}\oplus\mathfrak{k}_\sigma^\perp(\gamma)\right)\times\mathfrak{k}_\sigma(\gamma)}} \widehat{\mathrm{Tr}}_{\mathrm{s}}\Big[\mathrm{Ad}(k^{-1}\sigma)R_{Y_0^{\mathfrak{k}}}\big((f,Y^{\mathfrak{g}}),\mathrm{Ad}(k^{-1}\sigma)(f,Y^{\mathfrak{g}})\big)\Big] \mathrm{Tr}^E\Big[\rho^E(k^{-1}\sigma)\exp\big(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}})-A\big)\Big]\exp(-|Y_0^{\mathfrak{k}}|^2/2)\,\mathrm{d}y\,\mathrm{d}Y^{\mathfrak{g}}\,\mathrm{d}Y_0^{\mathfrak{k}}. \tag{6.30}$$

By (6.12), (6.30), we get (4.18). This completes the proof of Theorem 4.6.

7. Connections with the local equivariant index theory

In this section, we show that our formula in (4.18) is compatible with the local equivariant index theorems for compact locally symmetric spaces. We also apply our formula to study the twisted L_2-torsions introduced in [4] for compact locally symmetric spaces.

7.1. The classical Dirac operator on X

In this subsection, we will assume $\mathfrak{p}$ to be even dimensional and oriented, and K to be semisimple and simply connected. Recall that $m = \dim \mathfrak{p}$.

Let $\mathrm{Spin}(\mathfrak{p})$ be the Spin group of Euclidean space $\mathfrak{p}$, then the adjoint representation $K \to \mathrm{SO}(\mathfrak{p})$ lifts to a homomorphism $K \to \mathrm{Spin}(\mathfrak{p})$. Let $S^{\mathfrak{p}} = S_+^{\mathfrak{p}} \oplus S_-^{\mathfrak{p}}$ be the $\mathbb{Z}_2$-graded Hermitian vector space of $\mathfrak{p}$-spinors. To avoid confusion with the notation in Subsection 5.1, let $\bar{c}(\mathfrak{p})$ denote the Clifford algebra of $(\mathfrak{p}, B|_{\mathfrak{p}})$ acting on $S^{\mathfrak{p}}$. Therefore, K acts on $S_\pm^{\mathfrak{p}}$ via the spin representation $\rho^{S_\pm^{\mathfrak{p}}}$.

Set

$$P_{\mathrm{SO}}(X) = G \times_K \mathrm{SO}(\mathfrak{p}),\ P_{\mathrm{Spin}}(X) = G \times_K \mathrm{Spin}(\mathfrak{p}), \tag{7.1}$$

where K acts on $\mathrm{SO}(\mathfrak{p})$, $\mathrm{Spin}(\mathfrak{p})$ by conjugation. Then we get a double covering $P_{\mathrm{Spin}}(X) \to P_{\mathrm{SO}}(X)$, which defines a spin structure on X. Moreover, $S^{\mathfrak{p}}$ descends to the Hermitian vector bundle $S^{TX} = S_+^{TX} \oplus S_-^{TX}$ of

(TX, g^{TX})-spinors. Let $\nabla^{S^{TX}}$ denote the induced Clifford connection on S^{TX} by $\omega^{\mathfrak{k}}$.

We fix $\sigma \in \Sigma$, and we assume that its action on $\mathfrak{p}$ preserves the orientation. Then K^σ acts naturally on $P_{\mathrm{SO}}(X)$. We also assume that the homomorphism $K \to \mathrm{Spin}(\mathfrak{p})$ can be extended to a homomorphism $K^\sigma \to \mathrm{Spin}(\mathfrak{p})$, so that the action of K^σ on $P_{\mathrm{SO}}(X)$ lifts to an action on $P_{\mathrm{Spin}}(X)$. By [33, Definition 14.10 in Chapter 3], this is equivalent to say that K^σ preserves the spin structure.

Take (E, ρ^E) a unitary representation of K^σ. Now G^σ acts on $S^{TX} \otimes F$ over X preserving $\nabla^{S^{TX} \otimes F}$. Let D^X be the classical Dirac operator acting on $C^\infty(X, S^{TX} \otimes F)$. If $e_1, \ldots, e_m$ is an orthogonal basis of TX, then

$$D^X = \sum_{i=1}^m \bar{c}(e_i) \nabla_{e_i}^{S^{TX} \otimes F}. \tag{7.2}$$

Let $\mathcal{L}^X$ be the operator defined in (4.16), with E replaced by $S^{\mathfrak{p}} \otimes E$. Put

$$\mathcal{A} = -\frac{1}{48} \mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] - \frac{1}{2} C^{\mathfrak{k},E} \in \mathrm{End}(E). \tag{7.3}$$

It is clear that $\mathcal{A}$ commutes with K^σ. Then by [9, Theorem 7.2.1],

$$\frac{1}{2} D^{X,2} = \mathcal{L}^X_{\mathcal{A}}. \tag{7.4}$$

The integral kernel of $\exp(-tD^{X,2}/2)$, $t > 0$, lies in $\mathcal{Q}^\sigma$, so that, by taking the supertrace with respect to the $\mathbb{Z}_2$-grading of S^{TX}, the twisted orbital integral $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)]$ is well-defined.

Let $\gamma \in G$ be such that $\gamma\sigma$ is semisimple. We still assume that

$$\gamma = \mathrm{e}^a \, k^{-1}, a \in \mathfrak{p}, k \in K, \mathrm{Ad}(k)a = \sigma a. \tag{7.5}$$

Theorem 7.1. — *If $\gamma\sigma$ is non-elliptic, i.e., if $a \neq 0$, for $Y_0^{\mathfrak{k}} \in \mathfrak{k}_\sigma(\gamma)$,*

$$\mathrm{Tr_s}^{S^{\mathfrak{p}}} \left[\rho^{S^{\mathfrak{p}}}(k^{-1}\sigma) \exp(-\mathrm{i}\rho^{S^{\mathfrak{p}}}(Y_0^{\mathfrak{k}})) \right] = 0. \tag{7.6}$$

For any $t > 0$, we have

$$\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)] = 0. \tag{7.7}$$

Proof. — By [6, Proposition 3.23], we have

$$(-1)^{m/2} \Big(\mathrm{Tr_s}^{S^{\mathfrak{p}}} \left[\rho^{S^{\mathfrak{p}}}(k^{-1}\sigma) \exp(-\mathrm{i}\rho^{S^{\mathfrak{p}}}(Y_0^{\mathfrak{k}})) \right] \Big)^2 = \det\big(1 - \mathrm{Ad}(k^{-1}\sigma)\exp(-\mathrm{i}\,\mathrm{ad}(Y_0^{\mathfrak{k}}))\big)|_{\mathfrak{p}}. \tag{7.8}$$

If $a \neq 0$, then a is an eigenvector in $\mathfrak{p}$ of $\mathrm{Ad}(k^{-1}\sigma)\exp(-\mathrm{i}\,\mathrm{ad}(Y_0^{\mathfrak{k}}))$ associated with the eigenvalue 1, so that (7.6) holds.

Note that

$$\begin{aligned}(7.9)\quad \mathrm{Tr_s}^{S^{\mathfrak{p}}\otimes E}&\left[\rho^{S^{\mathfrak{p}}\otimes E}(k^{-1}\sigma)\exp\left(-\mathrm{i}\rho^{S^{\mathfrak{p}}\otimes E}(Y_0^{\mathfrak{k}})-t\mathcal{A}\right)\right]\\ &=\mathrm{Tr_s}^{S^{\mathfrak{p}}}\left[\rho^{S^{\mathfrak{p}}}(k^{-1}\sigma)\exp\left(-\mathrm{i}\rho^{S^{\mathfrak{p}}}(Y_0^{\mathfrak{k}})\right)\right]\\ &\qquad\cdot\mathrm{Tr}^{E}\left[\rho^{E}(k^{-1}\sigma)\exp\left(-\mathrm{i}\rho^{E}(Y_0^{\mathfrak{k}})-t\mathcal{A}\right)\right].\end{aligned}$$

Using Theorem 4.6, and combining (7.6) and (7.9), we get (7.7). □

7.2. The case of elliptic $\gamma\sigma$

We use the same assumptions as in Subsection 7.1. Now we fix an elliptic element $\gamma\sigma$, i.e. $\gamma = k^{-1} \in K$. Recall that $p = \dim \mathfrak{p}_\sigma(\gamma)$.

Recall that $N_{X(\gamma\sigma)/X}$ is the normal vector bundle of $X(\gamma\sigma)$ in X. Then

$$(7.10)\qquad TX|_{X(\gamma\sigma)} = TX(\gamma\sigma) \oplus N_{X(\gamma\sigma)/X}.$$

Note that

$$(7.11)\qquad \operatorname{rank} TX(\gamma\sigma) = p, \operatorname{rank} N_{X(\gamma\sigma)/X} = m - p.$$

Since we assume that σ preserves the orientation of $\mathfrak{p}$, both p and $m-p$ are even. Also the action of $\gamma\sigma$ on $TX|_{X(\gamma\sigma)}$ preserves the splitting in (7.10).

Let $\widehat{A}^{\gamma\sigma}(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}})$, $\widehat{A}^{\gamma\sigma}(N|_{X(\gamma\sigma)}, \nabla^{N|_{X(\gamma\sigma)}})$ be the equivariant $\widehat{A}$-genus given in [9, Subsection 7.7] for vector bundles $TX|_{X(\gamma\sigma)}$, $N|_{X(\gamma\sigma)}$. Note that there are questions of signs to be taken care of in these forms, we refer to [2, 3] and also [33, Theorem 14.11 in Chapter 3], [6, Chapter 6] for more detail. Let $o(TX(\gamma\sigma))$, $o(N_{X(\gamma\sigma)/X})$ be the orientation lines of $TX(\gamma\sigma)$, $N_{X(\gamma\sigma)/X}$ respectively. Because of the aforementioned sign ambiguity, $\widehat{A}^{\gamma\sigma}(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}})$ should be regarded as a section of $\Lambda^\bullet(T^*X(\gamma\sigma)) \otimes o(N_{X(\gamma\sigma)/X})$. Since $\mathfrak{p}$ is oriented by assumption, $\widehat{A}^{\gamma\sigma}(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}})$ can be viewed as a section of $\Lambda^\bullet(T^*X(\gamma\sigma)) \otimes o(TX(\gamma\sigma))$. A similar consideration can be made for $\widehat{A}^{\gamma\sigma}(N|_{X(\gamma\sigma)}, \nabla^{N|_{X(\gamma\sigma)}})$.

Let $\widehat{A}^{\gamma\sigma|_{\mathfrak{p}}}(0)$ be the degree 0 component of $\widehat{A}^{\gamma\sigma}(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}})$, and let $\widehat{A}^{\gamma\sigma|_{\mathfrak{k}}}(0)$ be the degree 0 component of $\widehat{A}^{\gamma\sigma}(N|_{X(\gamma\sigma)}, \nabla^{N|_{X(\gamma\sigma)}})$. These are constants on $X(\gamma\sigma)$. Put

$$(7.12)\qquad \widehat{A}^{\gamma\sigma}(0) = \widehat{A}^{\gamma\sigma|_{\mathfrak{p}}}(0) \cdot \widehat{A}^{\gamma\sigma|_{\mathfrak{k}}}(0).$$

The equivariant Chern character form of the bundle (F, ∇^F) is given by

$$(7.13)\qquad \mathrm{ch}^{\gamma\sigma}\left(F|_{X(\gamma\sigma)}, \nabla^{F|_{X(\gamma\sigma)}}\right) = \mathrm{Tr}\left[\rho^E(k^{-1}\sigma)\exp\left(-\frac{R^F|_{X(\gamma\sigma)}}{2\pi\mathrm{i}}\right)\right].$$

The above closed forms on $X(\gamma\sigma)$ are exactly the ones that appear in the Lefschetz fixed point formula of Atiyah–Bott [2, 3].

Let $\omega^{\mathfrak{z}_\sigma(\gamma)} = \omega^{\mathfrak{k}_\sigma(\gamma)} + \omega^{\mathfrak{p}_\sigma(\gamma)}$ be the left-invariant 1-form on $Z^0_\sigma(\gamma)$ valued in $\mathfrak{z}_\sigma(\gamma)$. Let $\Omega^{\mathfrak{z}_\sigma(\gamma)}$ be the curvature form of the connection form $\omega^{\mathfrak{k}_\sigma(\gamma)}$ on the principal bundle $Z^0_\sigma(\gamma) \to X(\gamma\sigma)$. As in (2.5), we have

$$\Omega^{\mathfrak{z}_\sigma(\gamma)} = -\frac{1}{2}[\omega^{\mathfrak{p}_\sigma(\gamma)}, \omega^{\mathfrak{p}_\sigma(\gamma)}] \in \Lambda^2(\mathfrak{p}_\sigma(\gamma)^*) \otimes \mathfrak{k}_\sigma(\gamma). \tag{7.14}$$

Then the curvatures R^F, R^{TX} restricting to $X(\gamma\sigma)$ are represented by the equivariant actions of $\Omega^{\mathfrak{z}_\sigma(\gamma)}$.

Using the same arguments as in the proof of [9, Proposition 7.1.1] and (7.14), we get the following identities of differential forms on $X(\gamma\sigma)$,

$$\begin{aligned} &\widehat{A}^{\gamma\sigma}\big(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}}\big)\widehat{A}^{\gamma\sigma}\big(N|_{X(\gamma\sigma)}, \nabla^{N|_{X(\gamma\sigma)}}\big) = \widehat{A}^{\gamma\sigma}(0). \\ &\mathrm{ch}^{\gamma\sigma}\left(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}}\right) + \mathrm{ch}^{\gamma\sigma}\left(N|_{X(\gamma\sigma)}, \nabla^{N|_{X(\gamma\sigma)}}\right) \\ &\qquad\qquad\qquad\qquad = \mathrm{Tr}^{\mathfrak{g}}\big[\mathrm{Ad}(k^{-1}\sigma)\big]. \end{aligned} \tag{7.15}$$

Let Ψ be the canonical element of norm 1 in $\Lambda^p(\mathfrak{p}_\sigma(\gamma)^*) \otimes o(\mathfrak{p}_\sigma(\gamma))$ (respectively, a section of norm 1 of $\Lambda^p(T^*X(\gamma\sigma)) \otimes o(TX(\gamma\sigma)))$. For $\alpha \in \Lambda^\bullet(\mathfrak{p}_\sigma(\gamma)^*) \otimes o(\mathfrak{p}_\sigma(\gamma))$ (respectively $\Lambda^\bullet(T^*X(\gamma\sigma)) \otimes o(TX(\gamma\sigma)))$, for $0 \leqslant l \leqslant p$, let $\alpha^{(l)}$ be the component of α of degree l. We define $\alpha^{\max} \in \mathbb{R}$ by

$$\alpha^{(p)} = \alpha^{\max}\Psi. \tag{7.16}$$

If $A \in \mathrm{End}(\mathfrak{p}_\sigma(\gamma))$ is antisymmetric, let $\mathrm{Pf}[A]$ be the Pfaffian of A. It is a polynomial function of A (with values twisted by $o(\mathfrak{p}_\sigma(\gamma))$), which is a square root of A. The form $\omega_A \in \Lambda^2(\mathfrak{p}^*_\sigma(\gamma))$ associated with A is given by $U, V \in \mathfrak{p}_\sigma(\gamma) \mapsto \langle U, AV\rangle$. Then

$$\mathrm{Pf}[A] = [\exp(\omega_A)]^{\max}. \tag{7.17}$$

Theorem 7.2. — *If $\gamma = k^{-1} \in K$, for any $t > 0$,*

$$\begin{aligned} &\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)] \\ &= \frac{1}{(2\pi t)^{p/2}} \int_{\mathfrak{k}_\sigma(\gamma)} J_{\gamma\sigma}(Y^{\mathfrak{k}}_0)\, \mathrm{Tr_s}^{S^{\mathfrak{p}}\otimes E}\Big[\rho^{S^{\mathfrak{p}}\otimes E}(k^{-1}\sigma) \\ &\qquad\qquad \cdot \exp\Big(-\mathrm{i}\rho^{S^{\mathfrak{p}}\otimes E}(Y^{\mathfrak{k}}_0) - t\mathcal{A}\Big)\Big] \\ &\qquad\qquad\qquad \cdot \exp(-|Y^{\mathfrak{k}}_0|^2/2t)\frac{\mathrm{d}Y^{\mathfrak{k}}_0}{(2\pi t)^{q/2}} \\ &= \Big[\widehat{A}^{\gamma\sigma}\big(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}}\big)\,\mathrm{ch}^{\gamma\sigma}\big(F|_{X(\gamma\sigma)}, \nabla^{F|_{X(\gamma\sigma)}}\big)\Big]^{\max}. \end{aligned} \tag{7.18}$$

Proof. — The first identity in (7.18) follows from Theorem 4.6 and (7.4). If (E, ρ^E) is an irreducible unitary representation of K^σ which is not irreducible when restricting to K, then by (2.5), (4.33) and (7.13), we get

$$\mathrm{ch}^{\gamma\sigma}(F|_{X(\gamma\sigma)}, \nabla^{F|_{X(\gamma\sigma)}}) = 0. \tag{7.19}$$

Then the second identity of (7.18) follows from (4.46).

We only need to prove the second identity in (7.18) for the case where (E, ρ^E) is irreducible for both groups K^σ and K. Recall that $K_\sigma(1) \subset K$ is just the fixed point set of σ action on K.

Since K is semisimple and simply connected, by [20, Lemma (3.15.4), Corollary (3.15.5)], $K_\sigma(1)$ is a connected subgroup of K, and there exists $v \in \mathfrak{k}_\sigma(1)$ such that v is regular in $\mathfrak{k}$. Then $\mathfrak{t} = \mathfrak{k}(v)$ is a Cartan subalgebra of $\mathfrak{k}$. Let T be the maximal torus of K corresponding to $\mathfrak{t}$, and let R^+ be the positive root system of (K, T) corresponding to the Weyl chamber of v. Then we get a decomposition of $\mathrm{Aut}(K)$ as in (4.35) with respect to (T, R^+). There exists $k_0 \in T$, $\tau \in \mathrm{Out}(K)$ such that the action of σ on K is given by $C(k_0) \circ \tau$. Moreover, $\mathfrak{s} = \mathfrak{t} \cap \mathfrak{k}_\sigma(1)$ is a Cartan subalgebra of $\mathfrak{k}_\tau(1)$, which is just the fixed point set of τ in $\mathfrak{t}$. Let S be the corresponded maximal torus of $K_\tau(1)$.

We extend $\tau \in \mathrm{Out}(K)$ to a $\widehat{\tau} \in \Sigma$, so that if $g \in G$, then

$$\widehat{\tau}(g) = k_0^{-1} \sigma(g) k_0 \in G. \tag{7.20}$$

Note that $\widehat{\tau}$ may not be of finite order. When acting on $\mathfrak{g}, \mathfrak{p}, \mathfrak{k}$, the adjoint action of $k^{-1}\sigma$ is the same as the adjoint action of $k^{-1}k_0\widehat{\tau}$. Following the same constructions as in the proof of Proposition 4.9, we may assume that E is an irreducible representation for both $K^{\hat{\tau}}$ and K. The group $K^{\hat{\tau}}$ also acts on $S^{\mathfrak{p}}$, so that the analogue of (4.37) holds. Then we will prove the second identity in (7.18) for $k^{-1}\widehat{\tau}$ instead of $k^{-1}\sigma$ with $k \in K$. By [43, Proposition I.4], and using the fact that the both sides of the second identity in (7.18) are invariant by conjugations of K, we can continue to assume that $k \in S$. Thus S is also a maximal torus of $K_{\hat{\tau}}(k^{-1})$.

Since (E, ρ^E) is an irreducible representation of K, then its highest weight $\lambda \in P_{++}$ is fixed by τ. Set

$$\rho_{\mathfrak{k}} = \frac{1}{2} \sum_{\alpha \in R^+} \alpha. \tag{7.21}$$

It is also fixed by τ. Then

$$\lambda,\ \rho_{\mathfrak{k}},\ \lambda + \rho_{\mathfrak{k}} \in \mathfrak{s}^*. \tag{7.22}$$

By [9, Proposition 7.5.2], we have

$$\mathcal{A} = 2\pi^2|\rho_{\mathfrak{k}} + \lambda|^2. \tag{7.23}$$

As in [9, (7.7.7)], if $y \in \mathfrak{k}_{\hat{\tau}}(k^{-1})$,

$$\begin{aligned}\operatorname{Tr_s}^{S^{\mathfrak{p}}}\Big[\rho^{S^{\mathfrak{p}}}(k^{-1}\widehat{\tau})\exp(-\mathrm{i}\bar{c}(\mathrm{ad}(y)))\Big] \\ = \mathrm{Pf}\big[\mathrm{ad}(y)|_{\mathfrak{p}_{\hat{\tau}}(k^{-1})}\big]\widehat{A}^{-1}\big(\mathrm{i}\,\mathrm{ad}(y)|_{\mathfrak{p}_{\hat{\tau}}(k^{-1})}\big)\Big(\widehat{A}^{\hat{\tau}^{-1}k\,\mathrm{e}^{\mathrm{i}y}}|_{\mathfrak{p}^{\perp}_{\hat{\tau}}(k^{-1})}(0)\Big)^{-1}.\end{aligned} \tag{7.24}$$

Then by (4.6), we have

$$\begin{aligned}J_{k^{-1}\widehat{\tau}}(y)\operatorname{Tr_s}^{S^{\mathfrak{p}}}\Big[\rho^{S^{\mathfrak{p}}}(k^{-1}\widehat{\tau})\exp\big(-\mathrm{i}\bar{c}(\mathrm{ad}(y))\big)\Big] \\ = (-1)^{\dim\mathfrak{p}^{\perp}_{\hat{\tau}}(k^{-1})/2}\,\mathrm{Pf}\big[\mathrm{ad}(y)|_{\mathfrak{p}_{\hat{\tau}}(k^{-1})}\big]\widehat{A}^{-1}\big(\mathrm{i}\,\mathrm{ad}(y)|_{\mathfrak{k}_{\tau}(k^{-1})}\big) \\ \widehat{A}^{\hat{\tau}^{-1}k}|_{\mathfrak{p}^{\perp}_{\hat{\tau}}(k^{-1})}(0)\left\{\frac{\det(1-\exp(-\mathrm{i}\,\mathrm{ad}(y))\,\mathrm{Ad}(k^{-1}\tau))|_{\mathfrak{k}^{\perp}_{\tau}(k^{-1})}}{\det(1-\mathrm{Ad}(k^{-1}\tau))|_{\mathfrak{k}^{\perp}_{\tau}(k^{-1})}}\right\}^{1/2}.\end{aligned} \tag{7.25}$$

Combining (4.18) with (7.25), we get

$$\begin{aligned}\operatorname{Tr_s}^{[k^{-1}\widehat{\tau}]}\big[\exp(-tD^{X,2}/2)\big] = \frac{(-1)^{\dim\mathfrak{p}^{\perp}_{\hat{\tau}}(k^{-1})/2}}{(2\pi t)^{p/2}}\,\mathrm{e}^{-2\pi^2 t|\lambda+\rho_{\mathfrak{k}}|^2} \\ \int_{\mathfrak{k}_{\tau}(k^{-1})}\mathrm{Pf}\big[\mathrm{ad}(y)|_{\mathfrak{p}_{\hat{\tau}}(k^{-1})}\big]\widehat{A}^{-1}\big(\mathrm{i}\,\mathrm{ad}(y)|_{\mathfrak{k}_{\tau}(k^{-1})}\big) \\ \widehat{A}^{\hat{\tau}^{-1}k}|_{\mathfrak{p}^{\perp}_{\hat{\tau}}(k^{-1})}(0)\left\{\frac{\det(1-\exp(-\mathrm{i}\,\mathrm{ad}(y))\,\mathrm{Ad}(k^{-1}\tau))|_{\mathfrak{k}^{\perp}_{\tau}(k^{-1})}}{\det(1-\mathrm{Ad}(k^{-1}\tau))|_{\mathfrak{k}^{\perp}_{\tau}(k^{-1})}}\right\}^{1/2} \\ \cdot\operatorname{Tr}^{E}\Big[\rho^{E}(k^{-1}\tau)\exp(-\mathrm{i}\rho^{E}(y))\Big]\exp(-|y|^2/2t)\frac{\mathrm{d}y}{(2\pi t)^{q/2}}.\end{aligned} \tag{7.26}$$

Let $\Omega^{\mathfrak{z}_{\hat{\tau}}(k^{-1})}$ be the curvature form associated with $Z^0_{\hat{\tau}}(k^{-1}) \to X(k^{-1}\widehat{\tau})$ as an analogue of Ω in (2.5), when replacing $\mathfrak{g}$ by $\mathfrak{z}_{\hat{\tau}}(k^{-1})$. In particular,

$$\Omega^{\mathfrak{z}_{\hat{\tau}}(k^{-1})} \in \Lambda^2\big(\mathfrak{p}_{\hat{\tau}}(k^{-1})^*\big) \otimes \mathfrak{k}_{\tau}(k^{-1}). \tag{7.27}$$

If $\alpha, \beta \in \Lambda^{\bullet}(\mathfrak{p}_{\hat{\tau}}(k^{-1})^*)$, $a, b \in \mathfrak{k}_{\tau}(k^{-1})$, we define

$$\langle \alpha\otimes a, \beta\otimes b\rangle' = \alpha\wedge\beta\langle a, b\rangle \in \Lambda^{\bullet}\big(\mathfrak{p}_{\hat{\tau}}(k^{-1})^*\big). \tag{7.28}$$

A direct calculation shows that

$$\big|\Omega^{\mathfrak{z}_{\hat{\tau}}(k^{-1})}\big|^{\prime,2} = 0. \tag{7.29}$$

By [9, (7.5.17)], we have

$$\mathrm{Pf}\big[\mathrm{ad}(y)|_{\mathfrak{p}_{\hat{\tau}}(k^{-1})}\big] = \Big[\exp\big(-\langle y, \Omega^{\mathfrak{z}_{\hat{\tau}}(k^{-1})}\rangle'\big)\Big]^{\max}. \tag{7.30}$$

Let $\Delta^{\mathfrak{k}_\tau(k^{-1})}$ and $\Delta^{\mathfrak{s}}$ be the standard (negative) Laplacian in $\mathfrak{k}_\tau(k^{-1})$ and $\mathfrak{s}$ respectively. Using the integral kernel of $\exp(t\Delta^{\mathfrak{k}_\tau(k^{-1})}/2)$ and (7.29), (7.30), we can rewrite (7.26) as follows,

$$(7.31)\quad \operatorname{Tr_s}^{[k^{-1}\widehat{\tau}]}\Big[\exp(-tD^{X,2}/2)\Big] = \frac{(-1)^{\dim \mathfrak{p}^{\perp}_{\hat\tau}(k^{-1})/2}}{(2\pi t)^{p/2}}\, e^{-2\pi^2 t|\lambda+\rho_{\mathfrak{k}}|^2}$$
$$\Bigg[\exp\left(t\Delta^{\mathfrak{k}_\tau(k^{-1})}/2\right)\Bigg\{\widehat{A}^{-1}\left(\mathrm{i}\,\mathrm{ad}(y)|_{\mathfrak{k}_\tau(k^{-1})}\right)$$
$$\widehat{A}^{\hat\tau^{-1}k|_{\mathfrak{p}^{\perp}_{\hat\tau}(k^{-1})}}(0)\left\{\frac{\det(1-\exp(-\mathrm{i}\,\mathrm{ad}(y))\,\mathrm{Ad}(k^{-1}\tau))|_{\mathfrak{k}^{\perp}_{\tau}(k^{-1})}}{\det(1-\mathrm{Ad}(k^{-1}\tau))|_{\mathfrak{k}^{\perp}_{\tau}(k^{-1})}}\right\}^{1/2}$$
$$\cdot \operatorname{Tr}^E\Big[\rho^E(k^{-1}\tau)\exp(-\mathrm{i}\rho^E(y))\Big]\Bigg\}\left(-t\Omega^{\mathfrak{z}_{\hat\tau}(k^{-1})}\right)\Bigg]^{\max}.$$

Let R' be the root system of $(\mathfrak{k}_\tau(k^{-1}),\mathfrak{s})$ and let R'_+ be a positive root system in R'. Let $\pi_{\mathfrak{k}_\tau(k^{-1})}(y)$, $y\in\mathfrak{s}$ be the functions defined as $\Pi_{\beta\in R'_+}\langle 2\pi i\beta, y\rangle$. Note that the function contained in $\{\cdots\}$ in (7.31) is invariant by adjoint action of $K_\tau(k^{-1})$. Then, we have

$$(7.32)\quad \exp(t\Delta^{\mathfrak{k}_\tau(k^{-1})}/2)\{\cdots\}(y)$$
$$= \frac{1}{\pi_{\mathfrak{k}_\tau(k^{-1})}(y)}\exp(t\Delta^{\mathfrak{s}}/2)\Big[\pi_{\mathfrak{k}_\tau(k^{-1})}(y)\{\cdots\}\Big](y).$$

The function in the right-hand side of (7.32) is viewed as a function in $y\in\mathfrak{s}$, which is invariant by the Weyl group $W(K^0_\tau(k^{-1}),S)$, and lifts to a central function on $\mathfrak{k}_\tau(k^{-1})$. Then it can be evaluated at $-t\Omega^{\mathfrak{z}_{\hat\tau}(k^{-1})}$.

If $\alpha\in R^+$, let $\mathfrak{k}_\alpha\subset\mathfrak{k}_{\mathbb{C}}$ be the associated root space. Put

$$(7.33)\qquad \mathfrak{n} = \sum_{\alpha\in R^+}\mathfrak{k}_\alpha.$$

Then τ preserves $\mathfrak{n}$. For $t\in T$, set $\delta(t\tau) = \det(1-\mathrm{Ad}(\tau^{-1}t^{-1}))|_{\mathfrak{n}}$. Note that up to multiplication by some constant, for $y\in\mathfrak{s}$, the analytic function $e^{2\pi\langle\rho_{\mathfrak{k}},y\rangle}\,\delta(e^{-\mathrm{i}y}\,k^{-1}\tau)$ coincides with

$$(7.34)\quad \pi_{\mathfrak{k}_\tau(k^{-1})}(y)\widehat{A}^{-1}\left(\mathrm{i}\,\mathrm{ad}(y)|_{\mathfrak{k}_\tau(k^{-1})}\right)$$
$$\times\Big[\det(1-\exp(-\mathrm{i}\,\mathrm{ad}(y))\,\mathrm{Ad}(k^{-1}\tau))|_{\mathfrak{k}^{\perp}_{\tau}(k^{-1})}\Big]^{1/2}.$$

A Weyl character formula for the non-connected compact Lie group K^τ was given in [20, Section 4.13, Proposition 4.13.1], we apply it to the K^τ-representation (E,ρ^E), we get that for $y\in\mathfrak{s}$,

$$
\begin{aligned}
(7.35)\quad & \mathrm{e}^{2\pi\langle\rho_{\mathfrak{k}},y\rangle}\,\delta(k^{-1}\tau\,\mathrm{e}^{-\mathrm{i}y})\,\mathrm{Tr}^E\left[k^{-1}\tau\exp(-\mathrm{i}y)\right] \\
&= \sum_{\omega\in W(\tau)} \det(\omega)\det\left(\mathrm{Ad}(\tau^{-1}k)\right)|_{\mathfrak{k}_{R_+\setminus\omega\cdot R_+}} \\
&\qquad\times \mathrm{Tr}\left[\rho^E(k^{-1}\tau)|_{E_{\omega\cdot\lambda}}\right]\mathrm{e}^{2\pi\langle\omega\cdot(\rho_{\mathfrak{k}}+\lambda),y\rangle},
\end{aligned}
$$

where $W(\tau)$ is a subgroup of Weyl group $W(K,T)$, and we refer to [20, Section 4.13] for the precise meaning of other notation. We only use this formula to make an observation that, by (7.22), (7.34) and (7.35), the function $\pi_{\mathfrak{k}_\tau(k^{-1})}(y)\{\cdots\}$, $y\in\mathfrak{s}$, in the right-hand side of (7.32), is an eigenfunction of $\Delta^{\mathfrak{s}}$ associated with the eigenvalue $2\mathcal{A}=4\pi^2|\rho_{\mathfrak{k}}+\lambda|^2$.

Then by (7.31), we get

$$
\begin{aligned}
(7.36)\quad & \mathrm{Tr_s}^{[k^{-1}\widehat{\tau}]}[\exp(-tD^{X,2}/2)] = \frac{1}{(2\pi t)^{p/2}} \\
&\times\left[\widehat{A}^{k^{-1}\widehat{\tau}}(0)\left(\widehat{A}^{k^{-1}\widehat{\tau}}\right)^{-1}\left(\mathrm{i}\,\mathrm{ad}(t\Omega^{\mathfrak{z}_{\widehat{\tau}}(k^{-1})})|_{\mathfrak{k}}\right)\mathrm{Tr}^E\left[k^{-1}\tau\,\mathrm{e}^{\mathrm{i}t\Omega^{\mathfrak{z}_{\widehat{\tau}}(k^{-1})}}\right]\right]^{\max}.
\end{aligned}
$$

Also the parameter t is killed automatically in the right-hand side of (7.36).

Note that the curvatures $R^{N|_{X(k^{-1}\widehat{\tau})}}$, $R^{TX|_{X(k^{-1}\widehat{\tau})}}$, $R^F|_{X(k^{-1}\widehat{\tau})}$ are given by the actions of the curvature form $\Omega^{\mathfrak{z}_{\widehat{\tau}}(k^{-1})}$ associated with $Z^0_{\widehat{\tau}}(k^{-1})\to X(k^{-1}\widehat{\tau})$. Then the right-hand side in (7.36) is just

$$
\begin{aligned}
(7.37)\quad & \Bigg[\widehat{A}^{k^{-1}\widehat{\tau}}(0)\left(\widehat{A}^{k^{-1}\widehat{\tau}}\right)^{-1}\left(N|_{X(k^{-1}\widehat{\tau})},\nabla^{N|_{X(k^{-1}\widehat{\tau})}}\right) \\
&\qquad\cdot\mathrm{Tr}^E\left[\rho^E(k^{-1}\tau)\exp\left(-\frac{R^F|_{X(k^{-1}\widehat{\tau})}}{2\pi\mathrm{i}}\right)\right]\Bigg]^{\max}
\end{aligned}
$$

Then by (7.13), (7.15), (7.37), we get the second identity in (7.18) for the semisimple element $k^{-1}\widehat{\tau}$. This completes the proof of our theorem. □

7.3. The local equivariant index theorem on Z

In this subsection, we make the same assumptions as in Subsections 2.6, 3.4 and 7.1. In particular, Γ is a cocompact torsion-free discrete subgroup of G such that $\sigma(\Gamma)=\Gamma$. Then $Z=\Gamma\backslash X$ is a compact manifold on which Σ^σ acts isometrically. The bundle S^{TX} descends to the bundle of TZ-spinors

S^{TZ}. The assumptions in Subsection 7.1 make $S^{TZ} \otimes F$ an equivariant Clifford module over Z equipped with the equivariant action of Σ^σ.

The operator D^X descends to the Dirac operator D^Z, which acts on $C^\infty(Z, S^{TZ} \otimes F)$ and commutes with Σ^σ, and the operator $\mathcal{L}^X_{\mathcal{A}}$ descends to $\mathcal{L}^Z_{\mathcal{A}}$. By (7.4),

$$\frac{1}{2} D^{Z,2} = \mathcal{L}^Z_{\mathcal{A}}. \tag{7.38}$$

Let $\ker D^Z \subset C^\infty(Z, S^{TZ} \otimes F)$ be the kernel of D^Z, which is a finite-dimensional representation of Σ^σ. The equivariant index of D^Z (or the Lefschetz number) associated with σ is defined as

$$\mathrm{Ind}_{\Sigma^\sigma}\left(\sigma, D^Z\right) = \mathrm{Tr_s}^{\ker D^Z}[\sigma]. \tag{7.39}$$

By McKean–Singer formula ([40], [6, Proposition 6.3]), for $t > 0$,

$$\mathrm{Ind}_{\Sigma^\sigma}\left(\sigma, D^Z\right) = \mathrm{Tr_s}\left[\sigma^Z \exp\left(-tD^{Z,2}/2\right)\right]. \tag{7.40}$$

Recall that ${}^\sigma Z \subset Z$ is the fixed point set of σ. By (2.95), it is a finite disjoint union of $[X(\gamma\sigma)]_Z$, $\underline{[\gamma]}_\sigma \in \underline{E}_\sigma$. Let $\widehat{A}^\sigma(TZ|_{{}^\sigma Z}, \nabla^{TZ|_{{}^\sigma Z}})$, $\mathrm{ch}^\sigma\left(F|_{{}^\sigma Z}, \nabla^{F|_{{}^\sigma Z}}\right)$ be the closed differential forms on ${}^\sigma Z$ defined as in Subsection 7.2.

By [2, 3] and [33, Theorem 14.11 in Chapter 3], $\mathrm{Ind}_{\Sigma^\sigma}(\sigma, D^Z)$ can be computed by the Lefschetz fixed point formula of Atiyah–Bott, so that

$$\mathrm{Ind}_{\Sigma^\sigma}\left(\sigma, D^Z\right) = \int_{{}^\sigma Z} \widehat{A}^\sigma\left(TZ|_{{}^\sigma Z}, \nabla^{TZ|_{{}^\sigma Z}}\right) \mathrm{ch}^\sigma\left(F|_{{}^\sigma Z}, \nabla^{F|_{{}^\sigma Z}}\right). \tag{7.41}$$

By Proposition 2.20, if $\underline{[\gamma]}_\sigma \in \underline{E}_\sigma$, the action of σ on $S^{TZ} \otimes F|_{[X(\gamma\sigma)]_Z}$ is equivalent to the action of $k^{-1}\sigma$ on the corresponding vector bundle $S^{TX} \otimes F$ over $\Gamma \cap Z(k^{-1}\sigma)\backslash X(k^{-1}\sigma)$. Then on each component $[X(\gamma\sigma)]_Z$ of ${}^\sigma Z$, the following function is constant,

$$\left[\widehat{A}^\sigma\left(TZ|_{{}^\sigma Z}, \nabla^{TZ|_{{}^\sigma Z}}\right) \mathrm{ch}^\sigma\left(F|_{{}^\sigma Z}, \nabla^{F|_{{}^\sigma Z}}\right)\right]^{\max} \tag{7.42}$$

and it is equal to

$$\left[\widehat{A}^{k^{-1}\sigma}\left(TX|_{X(k^{-1}\sigma)}, \nabla^{TX|_{X(k^{-1}\sigma)}}\right) \mathrm{ch}^{k^{-1}\sigma}\left(F|_{X(k^{-1}\sigma)}, \nabla^{F|_{X(k^{-1}\sigma)}}\right)\right]^{\max}. \tag{7.43}$$

Then by (3.45) and using Theorems 7.1, 7.2, we get

$$\begin{aligned} \mathrm{Tr_s}\left[\sigma^Z \mathrm{e}^{-tD^{Z,2}/2}\right] \\ = \sum_{\underline{[\gamma]}_\sigma \in \underline{E}_\sigma} \int_{[X(\gamma\sigma)]_Z} \widehat{A}^\sigma\left(TZ|_{{}^\sigma Z}, \nabla^{TZ|_{{}^\sigma Z}}\right) \mathrm{ch}^\sigma\left(F|_{{}^\sigma Z}, \nabla^{F|_{{}^\sigma Z}}\right). \end{aligned} \tag{7.44}$$

By (2.96) and (7.40), we see that (7.44) is equivalent to (7.41).

7.4. The de Rham operator associated with a flat bundle

From now on, we assume that G is a connected linear reductive Lie group with compact center. Then the center $\mathfrak{z}_{\mathfrak{g}}$ of $\mathfrak{g}$ is included in $\mathfrak{k}$. We do not assume anymore that K is semisimple or simply connected. We do not assume that σ preserves the orientation of $\mathfrak{p}$ either.

Put

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}, \ \mathfrak{u} = \sqrt{-1}\mathfrak{p} \oplus \mathfrak{k}. \tag{7.45}$$

Then $\mathfrak{u}$ is a real Lie algebra, which is called the compact form of $\mathfrak{g}$. It is clear that

$$\mathfrak{u}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}. \tag{7.46}$$

The form B extends to an invariant negative definite bilinear form on $\mathfrak{u}$ and to an invariant $\mathbb{C}$-bilinear form on $\mathfrak{g}_{\mathbb{C}}$. Let $G_{\mathbb{C}}$ be the connected group of complex matrices associated with $\mathfrak{g}_{\mathbb{C}}$, and let U be the analytic subgroup of $G_{\mathbb{C}}$ associated with $\mathfrak{u}$. Since G has compact center, by [28, Proposition 5.3], U is a compact Lie group. By [28, Proposition 5.6], $G_{\mathbb{C}}$ is still reductive, and G, U are closed subgroups of $G_{\mathbb{C}}$. In particular, U is a maximal compact subgroup of $G_{\mathbb{C}}$.

Let $U\mathfrak{u}$, $U\mathfrak{g}_{\mathbb{C}}$ be the enveloping algebras of $\mathfrak{u}$, $\mathfrak{g}_{\mathbb{C}}$ respectively. Then $U\mathfrak{g}_{\mathbb{C}}$ can be identified with the left-invariant holomorphic differential operators on $G_{\mathbb{C}}$. Let $C^{\mathfrak{u}}$ be the Casimir operator of U associated with B, by (4.9), we have

$$C^{\mathfrak{u}} = C^{\mathfrak{g}} \in U\mathfrak{g} \cap U\mathfrak{u}. \tag{7.47}$$

We extend the action σ to $\mathfrak{g}_{\mathbb{C}}$ as a complex linear isomorphism of $\mathfrak{g}_{\mathbb{C}}$. We assume that σ extends to an automorphism of U, then it also acts on $G_{\mathbb{C}}$ holomorphically. Set

$$U^{\sigma} = U \rtimes \Sigma^{\sigma}. \tag{7.48}$$

In the sequel, we fix a $(E, \rho^E) \in \mathrm{Irr}(U^{\sigma})$ with an invariant Hermitian metric h^E. By Weyl's unitary trick [28, Proposition 5.7], it extends uniquely to an irreducible representation of G^{σ}. We use the same notation ρ^E for the restrictions of this representation to G, to K and to K^{σ}. By (7.47), we have

$$C^{\mathfrak{u},E} = C^{\mathfrak{g},E} \in \mathrm{End}(E). \tag{7.49}$$

Put $F = G \times_K E$. Let ∇^F be the Hermitian connection induced by the connection form $\omega^{\mathfrak{k}}$. Then the map $(g, v) \in G \times_K E \to \rho^E(g)v \in E$ gives a canonical identification of vector bundles on X,

$$G \times_K E = X \times E. \tag{7.50}$$

Then F is equipped with a canonical flat connection $\nabla^{F,f}$ so that

$$\nabla^{F,f} = \nabla^F + \rho^E(\omega^{\mathfrak{p}}). \tag{7.51}$$

Let $(\Omega^\bullet_c(X,F), d^{X,F})$ be the (compactly supported) de Rham complex associated with $(F, \nabla^{F,f})$. Let $d^{X,F,*}$ be the adjoint operator of $d^{X,F}$ with respect to the L_2 metric on $\Omega^\bullet_c(X,F)$. The Dirac operator $\mathbf{D}^{X,F}$ of this de Rham complex is given by

$$\mathbf{D}^{X,F} = d^{X,F} + d^{X,F,*}. \tag{7.52}$$

As in (5.4), $c(TX)$, $\widehat{c}(TX)$ act on $\Lambda^\bullet(T^*X)$. We still use $e_1, \dots, e_m$ to denote an orthonormal basis of $\mathfrak{p}$ or TX, and let $\mathrm{e}^1, \dots, \mathrm{e}^m$ be the corresponding dual basis of $\mathfrak{p}^*$ or T^*X. Let $\nabla^{\Lambda^\bullet(T^*X)\otimes F,u}$ be the connection on $\Lambda^\bullet(T^*X) \otimes F$ induced by ∇^{TX} and ∇^F. Then the standard Dirac operator is given by

$$D^{X,F} = \sum_{j=1}^m c(e_j) \nabla^{\Lambda^\bullet(T^*X)\otimes F,u}_{e_j}. \tag{7.53}$$

By [12, (8.42)], we have

$$\mathbf{D}^{X,F} = D^{X,F} + \sum_{j=1}^m \widehat{c}(e_j)\rho^E(e_j). \tag{7.54}$$

Note that $C^{\mathfrak{g},E}$ defines an invariant parallel section of endomorphism of F. Recall that the operator $\mathcal{L}^X$ acting on $\Omega^\bullet(X,F)$ is defined as in (4.16). By [12, Proposition 8.4] and (4.15), (4.16), we have

$$\begin{aligned} \frac{\mathbf{D}^{X,F,2}}{2} &= \mathcal{L}^X - \frac{1}{2}C^{\mathfrak{g},E} - \frac{1}{8}B^*(\kappa^{\mathfrak{g}}, \kappa^{\mathfrak{g}}) \\ &= \frac{1}{2}C^{\mathfrak{g},X} - \frac{1}{2}C^{\mathfrak{g},E}. \end{aligned} \tag{7.55}$$

Moreover, $\mathbf{D}^{X,F,2}$ commutes with the action of G^σ.

The real rank (resp. complex rank) $\mathrm{rk}_{\mathbb{R}}\, G$ (resp. $\mathrm{rk}_{\mathbb{C}}\, G$) of G is defined as the real dimension of the maximal abelian subspace of $\mathfrak{p}$ (resp. the Cartan subalgebra of $\mathfrak{g}$). The fundamental rank of G is defined as

$$\delta(G) = \mathrm{rk}_{\mathbb{C}}\, G - \mathrm{rk}_{\mathbb{C}}\, K \in \mathbb{N}. \tag{7.56}$$

We still assume that $\gamma\sigma$ is a semisimple element given by (7.5). As explain in Remark 2.12, $Z^0_\sigma(\gamma)$ is real reductive equipped with a Cartan involution

$\theta|_{Z^0_\sigma(\gamma)}$. Let S be a maximal torus of $K^0_\sigma(\gamma)$ with Lie algebra $\mathfrak{s} \subset \mathfrak{k}_\sigma(\gamma)$. Set

$$\mathfrak{b}_\sigma(\gamma) = \{f \in \mathfrak{p}_\sigma(k^{-1}) \ : \ [f, \mathfrak{s}] = 0\}. \tag{7.57}$$

Then

$$a \in \mathfrak{b}_\sigma(\gamma), \ \dim_{\mathbb{R}} \mathfrak{b}_\sigma(\gamma) \geqslant \delta(Z^0_\sigma(\gamma)). \tag{7.58}$$

The quantity $\dim_{\mathbb{R}} \mathfrak{b}_\sigma(\gamma)$ only depends on the σ-conjugacy class of γ in G. If $\gamma\sigma$ is elliptic, then $\dim_{\mathbb{R}} \mathfrak{b}_\sigma(\gamma) = \delta(Z^0_\sigma(\gamma))$.

Let $e\big(TX(\gamma\sigma), \nabla^{TX(\gamma\sigma)}\big)$ be the Euler form of $TX(\gamma\sigma)$ associated with the Levi–Civita connection $\nabla^{TX(\gamma\sigma)}$. If $\dim \mathfrak{p}_\sigma(\gamma)$ is even, then

$$e\big(TX(\gamma\sigma), \nabla^{TX(\gamma\sigma)}\big) = \mathrm{Pf}\left[\frac{R^{TX(\gamma\sigma)}}{2\pi}\right]. \tag{7.59}$$

If $\dim \mathfrak{p}_\sigma(\gamma)$ is odd, then $e\big(TX(\gamma\sigma), \nabla^{TX(\gamma\sigma)}\big)$ vanishes identically.

Recall that the notation $[\cdot]^{\max}$ refers to the forms on $X(\gamma\sigma)$. The following theorem extends [9, Theorem 7.8.2].

Theorem 7.3. — *For $t > 0$, the following identity holds:*

$$\begin{aligned}\mathrm{Tr_s}^{[\gamma\sigma]}\big[\exp(-t\mathbf{D}^{X,F,2}/2)\big] &= \frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}} \exp\left(\frac{t}{8}B^*(\kappa^{\mathfrak{g}}, \kappa^{\mathfrak{g}})\right) \int_{\mathfrak{k}_\sigma(\gamma)} J_{\gamma\sigma}(Y_0^{\mathfrak{k}}) \\ &\quad \mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)\otimes E}\left[\rho^{\Lambda^\bullet(\mathfrak{p}^*)\otimes E}(k^{-1}\sigma) \exp\big(-\mathrm{i}\rho^{\Lambda^\bullet(\mathfrak{p}^*)\otimes E}(Y_0^{\mathfrak{k}}) + \frac{t}{2}C^{\mathfrak{g},E}\big)\right] \\ &\quad \exp\big(-|Y_0^{\mathfrak{k}}|^2/2t\big) \frac{\mathrm{d}Y_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.\end{aligned} \tag{7.60}$$

If $\dim \mathfrak{b}_\sigma(\gamma) \geqslant 1$, *then*

$$\mathrm{Tr_s}^{[\gamma\sigma]}\big[\exp(-t\mathbf{D}^{X,F,2}/2)\big] = 0. \tag{7.61}$$

If $\gamma\sigma$ is elliptic, then

$$\mathrm{Tr_s}^{[\gamma\sigma]}\big[\exp(-t\mathbf{D}^{X,F,2}/2)\big] = \Big[e\big(TX(\gamma\sigma), \nabla^{TX(\gamma\sigma)}\big)\Big]^{\max} \mathrm{Tr}^E\left[\rho^E(\gamma\sigma)\right]. \tag{7.62}$$

Proof. — The identity in (7.60) follows from (4.15), (4.18), (7.55). As in (7.9), the integrand in (7.60) contains the following factor

$$\mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)}\left[\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(k^{-1}\sigma)\, \mathrm{e}^{-\mathrm{i}\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(Y_0^{\mathfrak{k}})}\right] = \det\big(1 - \exp(\mathrm{i}\,\mathrm{ad}(Y_0^{\mathfrak{k}}))\,\mathrm{Ad}(\sigma^{-1}k)\big)|_{\mathfrak{p}}. \tag{7.63}$$

If $\dim \mathfrak{b}_\sigma(\gamma) \geqslant 1$, then the right-hand side in (7.63) vanishes identically for $Y_0^{\mathfrak{k}} \in \mathfrak{k}_\sigma(\gamma)$. Then (7.61) follows.

Now take $\gamma = k^{-1} \in K$. Then

$$\mathfrak{b}_\sigma(\gamma) \subset \mathfrak{p}_\sigma(\gamma). \tag{7.64}$$

Moreover, by [28, p. 129], $\mathfrak{b}_\sigma(\gamma) \oplus \mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{z}_\sigma(\gamma)$. In this case, $\dim \mathfrak{p}_\sigma(\gamma) - \dim \mathfrak{b}_\sigma(\gamma)$ is even. Note that $\Omega^{\mathfrak{z}_\sigma(\gamma)}$ is the curvature form given in (7.14).

By (7.14), as an analogue of (7.15), we have the following identities

$$\begin{gathered}\widehat{A}^{-1}\big(\mathrm{i}\,\mathrm{ad}(-t\Omega^{\mathfrak{z}_\sigma(\gamma)})|_{\mathfrak{z}_\sigma(\gamma)}\big) \\ \times\left[\frac{\det(1-\mathrm{e}^{-\mathrm{i}\,\mathrm{ad}(-t\Omega^{\mathfrak{z}_\sigma(\gamma)})}\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}_\sigma^{\perp}(\gamma)}}{\det(1-\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}_\sigma^{\perp}(\gamma)}}\right]^{1/2} = 1, \\ \mathrm{Tr}^E\left[\rho^E(k^{-1}\sigma)\exp(-\mathrm{i}\rho^E(-t\Omega^{\mathfrak{z}_\sigma(\gamma)}))\right] = \mathrm{Tr}^E\left[\rho^E(k^{-1}\sigma)\right].\end{gathered} \tag{7.65}$$

Note that if $\dim \mathfrak{b}_\sigma(\gamma) \geqslant 1$, if $Y_0^{\mathfrak{k}} \in \mathfrak{k}_\sigma(\gamma)$, then

$$\mathrm{Pf}\left[\mathrm{ad}(Y_0^{\mathfrak{k}})\right] = 0. \tag{7.66}$$

By (7.14), (7.59), (7.66), we get that $e\big(TX(\gamma\sigma), \nabla^{TX(\gamma\sigma)}\big) = 0$. Then (7.61) is compatible with (7.62). We only need to consider the case where $\dim \mathfrak{b}_\sigma(\gamma) = 0$, so that $\mathfrak{s}$ is also a Cartan subalgebra of $\mathfrak{z}_\sigma(\gamma)$.

If we make the same assumptions on K, $\mathfrak{p}$ and σ as in Subsection 7.1, then (7.62) is a special case of Theorem 7.2. In general, we can proceed as in the proof of Theorem 7.2 with the group U instead of K. Note that the Lie algebra of $\mathrm{Aut}(U)$ is isomorphic to $[\mathfrak{u},\mathfrak{u}]$. By [20, Lemma (3.15.4)], if $\sigma \in \mathrm{Aut}(U)$, then $[\mathfrak{u},\mathfrak{u}](\sigma)$ contains regular elements in $[\mathfrak{u},\mathfrak{u}]$, so that there always exists $v \in \mathfrak{u}(\sigma) \cap \mathfrak{u}^{\mathrm{reg}}$. Then we fix the corresponding maximal torus and a positive root system R^+ for U as in the proof of Theorem 7.2. Let $\rho_{\mathfrak{u}}$ denote the element defined as in (7.21).

We may suppose that (E, ρ^E) is irreducible for both U and U^σ, so that $C^{\mathfrak{g},E}$ is scalar. Let λ be the highest weight for this U-representation. By [9, Proposition 7.5.2], we have

$$-C^{\mathfrak{g},E} - \frac{1}{4}B^*(\kappa^{\mathfrak{g}}, \kappa^{\mathfrak{g}}) = 4\pi^2|\rho_{\mathfrak{u}} + \lambda|^2. \tag{7.67}$$

Based on the above constructions, the arguments in the proof of Theorem 7.2 still work without assuming U to be semisimple or simply connected. Using instead (7.65) and (7.67), we can prove (7.62) in full generality. This completes the proof of our theorem. □

7.5. Twisted L_2-torsion

Following the idea in last subsection, our formula for twisted orbital integrals is quite promising in studying the equivariant real analytic torsions for compact locally symmetric spaces. We refer to another publication of the author [36] for a detailed investigation on this topic. Here, we give a brief discussion on the twisted L_2-torsion introduced by Bergeron and Lipnowski [4].

We make the same assumptions as in Subsections 2.6, 3.4 and 7.4. In particular, G is linear reductive and with compact center, Γ is a cocompact torsion-free discrete subgroup of G such that $\sigma(\Gamma) = \Gamma$. Let (E, ρ^E) be an irreducible unitary representation for both U and U^σ. Furthermore, we make an assumption on the representation (E, ρ^E): as G-representations, $(E, \rho^E) \not\cong (E, \rho^E \circ \theta)$. By [14, §VI, Theorem 5.3] and [5, Lemma 4.1], the flat vector bundle $F \to Z = \Gamma\backslash X$ is (strongly) acyclic.

Let $\overline{\sigma} \in \mathrm{Aut}(\Gamma)$ be the induced isomorphism of σ. Then $\overline{\sigma}$ is of finite order $N_0 \in \mathbb{N}^*$ (since Γ is always finitely generated).

Lemma 7.4. — *The action of σ^{N_0} on X is the identity map. Then for $\gamma \in \Gamma$, $\gamma\sigma$ is elliptic if and only if $(\gamma\overline{\sigma})^{N_0} = 1$.*

Proof. — The first statement is equivalent to that σ^{N_0} acts on $\mathfrak{p}$ as identity. In fact, if a nonzero $a \in \mathfrak{p}$ is such that $a' = \sigma^{N_0}(a)$, then the function $t \in \mathbb{R} \mapsto d(p\,\mathrm{e}^{ta}, p\,\mathrm{e}^{ta'})$ is either constant 0 or tending to infinity as $t \to +\infty$ (cf. [21, Proposition 1.4.1, p. 19]). For any $\gamma \in \Gamma$, we have $d(p\,\mathrm{e}^{ta}, \gamma) = d(p\,\mathrm{e}^{ta'}, \gamma)$ since σ^{N_0} fixes γ. Then $d(p\,\mathrm{e}^{ta}, p\,\mathrm{e}^{ta'}) \leqslant 2d(p\,\mathrm{e}^{ta}, \gamma)$, the cocompactness of Γ infers that $d(p\,\mathrm{e}^{ta}, p\,\mathrm{e}^{ta'})$ is bounded hence must be constant 0, we conclude that $a = a' = \sigma^{N_0}(a)$.

For $\gamma \in \Gamma$, $\gamma\sigma$ is semisimple, we can take $g \in G$ such that $\sigma^{N_0}(g) = g$, and $\gamma = g\,\mathrm{e}^a\,k^{-1}\sigma(g^{-1})$ where $a \in \mathfrak{p}$, $k \in K$ and $\mathrm{Ad}(k)a = \sigma(a)$. Then the second part follows directly from

$$(\gamma\sigma)^{N_0} = g\,\mathrm{e}^{N_0 a}\,k^{-1}\sigma(k^{-1})\cdots\sigma^{N_0-1}(k^{-1})g^{-1}\sigma^{N_0}. \qquad \square$$

Recall that $\underline{E}_\sigma$ denotes the finite set of elliptic classes in $[\Gamma]_\sigma = [\Gamma]_{\overline{\sigma}}$. Set $Z^1(\overline{\sigma}, \Gamma) = \{\gamma \in \Gamma \;:\; (\gamma\overline{\sigma})^{N_0} = 1 \in \Gamma\}$, and let $H^1(\overline{\sigma}, \Gamma)$ denote the quotient of $Z^1(\overline{\sigma}, \Gamma)$ by the equivalent relation defined by the σ-conjugation by elements in Γ. The above lemma implies the identification

$$H^1(\overline{\sigma}, \Gamma) = \underline{E}_\sigma. \tag{7.68}$$

Let $N^{\Lambda^\bullet(\mathfrak{p}^*)}$, $N^{\Lambda^\bullet(T^*X)}$ be the number operators of $\Lambda^\bullet(\mathfrak{p}^*)$, $\Lambda^\bullet(T^*X)$. For $\underline{[\gamma]_\sigma} \in \underline{E}_\sigma$, $t>0$, set

$$\mathcal{E}_{X,\gamma\sigma}(F,t) = \mathrm{Tr_s}^{[\gamma\sigma]}\left[\left(N^{\Lambda^\bullet(T^*X)} - \frac{m}{2}\right)\exp\left(-t\mathbf{D}^{X,F,2}/2\right)\right]. \tag{7.69}$$

Since $\gamma\sigma$ is elliptic, there exist $g\in G$ such that $k = g\gamma\sigma(g^{-1})\in K$. Let λ still denote the highest weight for the U-representation ρ^E as in the proof of Theorem 7.3. By (4.18), (7.67), we have

$$\begin{aligned}\mathcal{E}_{X,\gamma\sigma}(F,t) = \frac{1}{(2\pi t)^{p/2}}\exp\left(-2\pi^2 t|\lambda+\rho_{\mathfrak{u}}|^2\right)\\ \cdot\int_{\mathfrak{k}_\sigma(k)} J_{k\sigma}(Y_0^{\mathfrak{k}})\,\mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)}\left[\left(N^{\Lambda^\bullet(\mathfrak{p}^*)} - \frac{m}{2}\right)\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(k\sigma)\,\mathrm{e}^{-\mathrm{i}\rho^{\Lambda^\bullet(\mathfrak{p}^*)}(Y_0^{\mathfrak{k}})}\right]\\ \mathrm{Tr}^E\left[\rho^E(k\sigma)\exp(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}}))\right]\mathrm{e}^{-|Y_0^{\mathfrak{k}}|^2/2t}\,\frac{\mathrm{d}Y_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.\end{aligned} \tag{7.70}$$

Set

$$\mathrm{Tr_s}^{\Gamma,\prime}\left[\sigma^Z\,\mathrm{e}^{-t\mathbf{D}^{Z,F,2}/2}\right] = \sum_{\underline{[\gamma]_\sigma}\in\underline{E}_\sigma}\mathrm{Vol}(\Gamma\cap Z_\sigma(\gamma)\backslash X(\gamma\sigma))\mathcal{E}_{X,\gamma\sigma}(F,t), \tag{7.71}$$

where the prime ($'$) refers to the number operator $N^{\Lambda^\bullet(\mathfrak{p}^*)}$ involved.

Proposition 7.5. — *If $\delta(Z_\sigma^0(\gamma))\neq 1$, then for $t>0$,*

$$\mathcal{E}_{X,\gamma\sigma}(F,t) = 0. \tag{7.72}$$

For non-vanishing cases, there exists a constant $C>0$ such that for $t\in\,]0,1]$

$$\begin{aligned}\left|\sqrt{t}\mathcal{E}_{X,\gamma\sigma}(F,t)\right| \leqslant C,\\ \left|\left(1+2t\frac{\partial}{\partial t}\right)\mathcal{E}_{X,\gamma\sigma}(F,t)\right| \leqslant C\sqrt{t}.\end{aligned} \tag{7.73}$$

Then, as $t\to 0$, $\mathcal{E}_{X,\gamma\sigma}(F,t)$ has the asymptotic expansion in the form of

$$\frac{1}{\sqrt{t}}\sum_{j=0}^{+\infty} a_j^{\gamma\sigma}t^j,\ \text{with } a_j^{\gamma\sigma}\in\mathbb{C} \text{ for } j\in\mathbb{N}. \tag{7.74}$$

There exist constants $C'>0$, $c'>0$ such that for $t\gg 0$, we have

$$\left|\mathcal{E}_{X,\gamma\sigma}(F,t)\right| \leqslant C'\,\mathrm{e}^{-c't}. \tag{7.75}$$

Proof. — Note that (7.72) was proved in [36, Proposition 3.3.3 and Corollary 3.3.4], it follows from the identities as in (4.59). The estimates (7.73) were proved in [36, (4.4.5) in Theorem 4.4.1], and as a consequence, we get the asymptotic expansion (7.75).

In the context of cyclic base change with $\gamma = 1$, the estimate (7.75) was proved in [4, Lemma 4.10]. For general setting as here, it was proved in [36, (4.4.6) in Theorem 4.4.1]. Note that for this conclusion, the assumption on $\rho^E \circ \theta$ is crucial (called a nondegeneracy condition for ρ^E). □

Definition 7.6. — *We define the σ-twisted L_2-torsion for $Z = \Gamma\backslash X$ associated with the flat vector bundle F as follows,*

$$\mathcal{T}_{\sigma,L_2}(Z,F) = -\frac{1}{2}\int_0^{+\infty}\left(1+2t\frac{\partial}{\partial t}\right)\mathrm{Tr_s}^{\Gamma,\prime}\left[\sigma^Z\,\mathrm{e}^{-t\mathbf{D}^{Z,F,2}/2}\right]\frac{\mathrm{d}t}{t}. \tag{7.76}$$

By Proposition 7.5 and (7.71), $\mathcal{T}_{\sigma,L_2}(Z,F)$ is well-defined as a number. In particular, only the elliptic class $\underline{[\gamma]}_\sigma$ such that $\delta(Z^0_\sigma(\gamma)) = 1$ contributes to $\mathcal{T}_{\sigma,L_2}(Z,F)$. If there is no such elliptic class, we get $\mathcal{T}_{\sigma,L_2}(Z,F) = 0$.

Example 7.7. — As in [4], assume that σ has finite order, and that $H^1(\sigma,G) = 1$ (which implies that for $\gamma \in H^1(\sigma,\Gamma)$, it is σ-conjugate to 1 by elements in G). Recall that ${}^\sigma Z \subset Z$ is the fixed point set of σ in Z, then

$$\begin{aligned}\mathrm{Tr_s}^{\Gamma,\prime}&\left[\sigma^Z\,\mathrm{e}^{-t\mathbf{D}^{Z,F,2}/2}\right]\\ &= \mathrm{Vol}({}^\sigma Z)\,\mathrm{Tr_s}^{[\sigma]}\left[\left(N^{\Lambda^\bullet(T^*X)}-\frac{m}{2}\right)\exp\left(-t\mathbf{D}^{X,F,2}/2\right)\right].\end{aligned} \tag{7.77}$$

By [4, Theorem 4.11], $\mathcal{T}_{\sigma,L_2}(Z,F)$ appeared as the limit of the σ-equivariant analytic torsions under a tower of finite coverings of $Z = \Gamma\backslash X$.

As explained in Subsection 4.5, if we write further the twisted orbital integral $\mathrm{Tr_s}^{[\sigma]}[\cdots]$ in terms of the ordinary identity orbital integral associated with the subgroup $Z_\sigma(1)$, fixed point set of σ in G. Then $\mathcal{T}_{\sigma,L_2}(Z,F)$ is actually a linear combination of the ordinary L_2-torsions ([37, 39]) of ${}^\sigma Z$.

Example 7.8. — In [11, 12], Bismut, Ma and Zhang showed that for a universally constructed sequence of flat vector bundles F_d, $d \in \mathbb{N}$ over a closed manifold Z, under the nondegeneracy condition, as $d \to +\infty$,

$$\mathcal{T}(Z,F_d) = \mathcal{T}_{L_2}(Z,F_d) + \mathcal{O}(\mathrm{e}^{-cd}), \tag{7.78}$$

where $\mathcal{T}(Z,F_d)$, $\mathcal{T}_{L_2}(Z,F_d)$ denote the real analytic torsions, L_2-torsions respectively. In the context of a locally symmetric space, a new proof of (7.78) using Selberg trace formula was given in [42].

In [36, Section 4], the author considered the asymptotic expansion of σ-equivariant analytic torsions $\mathcal{T}_\sigma(Z,F_d)$ as $d \to +\infty$ for the compact locally symmetric space $Z = \Gamma\backslash X$. We fix a nondegenerate unitary representation $(E,\rho^E) \in \mathrm{Irr}(U^\sigma)\cap\mathrm{Irr}(U)$ which has the highest weight λ. Associated with it, in [36, Subsection 4.2], the author constructed a canonical sequence

$(E_d, \rho^{E_d}) \in \mathrm{Irr}(U^\sigma) \cap \mathrm{Irr}(U), d \in \mathbb{N}^*$, such that ρ^{E_d} has highest weight $d\lambda$. By [36, Proposition 4.6.1], as $d \to +\infty$,

$$\mathcal{T}_\sigma(Z, F_d) = \mathcal{T}_{\sigma,L_2}(Z, F_d) + \mathcal{O}(\mathrm{e}^{-cd}), \tag{7.79}$$

The main result of [36, Section 4] showed that the leading term (in d) of $\mathcal{T}_{\sigma,L_2}(Z, F_d)$ is given in terms of W-invariants, for the fixed point set ${}^\sigma Z$, introduced in [11, 12].

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